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Geometry of slow–fast Hamiltonian systems and Painlevé equations

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Highlights

- Geometric tools to describe slow–fast Hamiltonian systems on smooth manifolds.
- Direct derivation of Painlevè-1 equation as a principal part of a slow–fast Hamiltonian system near a fold point of a slow manifold.
- Direct derivation of Painlevè-2 equation as a principal part of a slow–fast Hamiltonian system near a cusp point of a slow manifold.

Abstract

In the first part of the paper we introduce some geometric tools needed to describe slow–fast Hamiltonian systems on smooth manifolds. We start with a smooth bundle $p : M \to B$ where (M, ω) is a C^∞-smooth presymplectic manifold with a closed constant rank 2-form ω and (B, λ) is a smooth symplectic manifold. The 2-form ω is supposed to be compatible with the structure of the bundle, that is the bundle fibers are symplectic manifolds with respect to the 2-form ω and the distribution on M generated by kernels of ω is transverse to the tangent spaces of the leaves and the dimensions of the kernels and of the leaves are supplementary. This allows one to define a symplectic structure $\Omega_\varepsilon = \omega + \varepsilon^{-1} p^*\lambda$ on M for any positive small ε, where $p^*\lambda$ is the lift of the 2-form λ to M. Given a smooth Hamiltonian H on M one gets a slow–fast Hamiltonian system with respect to Ω_ε. We define a slow manifold SM for this system. Assuming SM is a smooth submanifold, we define a slow Hamiltonian flow on SM. The second part of the paper deals with singularities of the restriction of p to SM. We show that if $\dim M = 4$, $\dim B = 2$ and Hamilton function H is generic, then the behavior of the system near a singularity of fold type is described,

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to the main order, by the equation Painlevé-I, and if this singularity is a cusp, then the related equation is Painlevé-II.

Keywords: Slow–fast; Hamiltonian; Presymplectic manifold; Singular symplectic; Bundle; Disruption point; Blow-up; Painlevé equations

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1. Introduction

Q3 Slow–fast Hamiltonian systems are ubiquitous in the applications in different fields of science. These applications range from astrophysics, plasma physics and ocean hydrodynamics to molecular dynamics. Usually these problems are given in coordinate form, moreover, in the form where a symplectic structure in the phase space is standard (in Darboux coordinates). But there are cases when either the symplectic form is nonstandard or the system under study is of a kind where the corresponding symplectic form has to be found, in particular, when we deal with the system on a manifold.

It is our aim in this paper to present basic geometric tools to describe slow–fast Hamiltonian systems on manifolds, that is in a coordinate-free way. For the non Hamiltonian case this was done by V.I. Arnold [1]. Recall that a customary slow–fast dynamical system is defined by a system of differential equations

$$\varepsilon \dot{x} = f(x, y, \varepsilon), \qquad \dot{y} = g(x, y, \varepsilon), \quad (x, y) \in \mathbb{R}^m \times \mathbb{R}^n, \tag{1}$$

depending on a small positive parameter ε (its positivity is needed to fix the direction of increasing time t). It is evident that x-variables in the region of the phase space where $f \neq 0$ change with the speed $\sim 1/\varepsilon$ that is fast. In comparison with them the change of y-variables is slow. Therefore variables x are called fast and y are called slow.

Such system generates two limiting systems whose properties influence the dynamics of the slow–fast system for a small ε. One of the limiting system is called fast or layer system and is derived in the following way. Let us introduce the so-called fast time $\tau = t/\varepsilon$. Then the system acquires the parameter ε in the right hand side of the second equation (due to the differentiation in τ) but looses it in the first equation. Thus, the right hand sides depend on ε in a regular way

$$\frac{dx}{d\tau} = f(x, y, \varepsilon), \qquad \frac{dy}{d\tau} = \varepsilon g(x, y, \varepsilon), \quad (x, y) \in \mathbb{R}^m \times \mathbb{R}^n. \tag{2}$$

Setting then $\varepsilon = 0$ we get the system, where y-variables are constants $y = y_0$ and they can be considered as parameters in the equations for x. Sometimes these equations are called layer equations. Because the fast system depends on parameters, it may pass through many bifurcations as parameters y change and this can be useful to find some special motions in the full system for small $\varepsilon > 0$.

The slow equations are derived as follows. Let us formally set $\varepsilon = 0$ in the system (1) and solve the equations $f = 0$ with respect to x (where it is possible). The most natural case when this can be done, is when the matrix f_x is invertible in some domain where solutions for equations $f = 0$ exist. Then by the implicit function theorem one can solve the system $f = 0$. Denote the related branch of solutions as $x = h(y)$ and insert it into the second equation instead of x. Then one gets a system of differential equations for y variables

$$\dot{y} = g(h(y), y, 0),$$

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