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Orbit space reduction and localizations

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Abstract

We review the familiar method of reducing a symmetric ordinary differential equation via invariants of the symmetry group. Working exclusively with polynomial invariants is problematic: Generator systems of the polynomial invariant algebra, as well as generator systems for the ideal of their relations, may be prohibitively large, which makes reduction unfeasible. In the present paper we propose an alternative approach which starts from a characterization of common invariant sets of all vector fields with a given symmetry group, and uses suitably chosen localizations. We prove that there exists a reduction to an algebraic variety of codimension at most two in its ambient space. Some examples illustrate the approach.

Keywords: Symmetry; Invariant theory; Algebraic group

1. Introduction

In this note we consider a symmetric ordinary differential equation

$$\dot{x} = f(x) \tag{1}$$

on $\mathbb{R}^n$ or $\mathbb{C}^n$, with the symmetries forming a subgroup G of the general linear group $GL(n)$. There is an obvious benefit to be gained from symmetries, as one may employ them to find new solutions from given ones. Moreover there is a less obvious (but also familiar) benefit, since one may employ symmetry reduction. As representatives of the many publications on the subject we mention only Field [7] (for compact groups), and Cushman and Bates [4] (for reductive groups).

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In the present paper we first focus on the conceptually straightforward approach to reduction via the Hilbert map which is constructed from polynomial invariants of G. (To avoid further technicalities we will discuss only polynomial vector fields here.) The survey article by Chossat [3] provides a very good introduction to this method for compact G, as well as a number of examples. Chossat also points out the limitations of the approach: There may be problems with its feasibility, since (even minimal) generator systems of polynomial invariants may be very large. The main purpose of the present paper is to suggest a possible escape from such feasibility problems. Roughly speaking, we will introduce a refinement of orbit space reduction via polynomials by introducing carefully chosen denominators to achieve a reduction via rational functions. Using a theorem due to Grosshans [8], we show that the necessary number of generators is at most two higher than the number dictated by dimensions of group orbits. In our proofs we will make use of some results and tools from commutative algebra and elementary algebraic geometry. A few examples illustrate the reduction method.

2. An overview of the orbit space method

2.1. Blanket assumptions and notation

We first introduce some notions and hypotheses which will be kept throughout the paper. Some of the assumptions as stated are more restrictive than necessary; our focus is on reduction mechanisms and we want to keep technicalities to a minimum.

- $\mathbb{K}$ stands as an abbreviation for $\mathbb{R}$ or $\mathbb{C}$.
- $G \subseteq GL(n, \mathbb{K})$ is a linear algebraic group (i.e. a subgroup of $GL(n, \mathbb{K})$ defined by polynomial equations) which acts naturally on $\mathbb{K}^n$.
- Furthermore the orbits of G have generic dimension $s > 0$.
- We restrict attention to the case of polynomial vector fields, thus f has polynomial entries (unless specified otherwise).
- The differential equation (1) is symmetric (equivariant) with respect to G; i.e., $T^{-1} f T = f$ for all $T \in G$.

Recall that a polynomial $\psi \in \mathbb{K}[x_1, \ldots, x_n]$ is called G-invariant if $\psi \circ T = \psi$ for all $T \in G$. The G-invariant polynomials form a subalgebra of the polynomial algebra which is denoted $\mathbb{K}[x_1, \ldots, x_n]^G$. We add one more hypothesis.

- We assume that $\mathbb{K}[x_1, \ldots, x_n]^G \neq \mathbb{K}$ admits a finite set $\gamma_1, \ldots, \gamma_r$ of generators.
- Accordingly we define the Hilbert map

$$\Gamma := \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_r \end{pmatrix} : \mathbb{K}^n \to \mathbb{K}^r.$$

The choice of G as a linear group is not as restrictive as it may initially seem: There are local linearization theorems for compact groups (Bochner; see e.g. Duistermaat and Kolk [6]) and for semisimple Lie groups (see e.g. Kushnirenko [13]). The choice of f as a polynomial is restrictive, but one may think of Taylor expansions and results e.g. by Schwarz [22], Luna [15] and Poénaru [18] which guarantee extensions to the smooth and analytic case.

2.2. The basic reduction mechanism

There is one more notion we need to introduce.

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