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The dynamics of a fold-and-twist map

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Abstract

Inspired by a discrete-time predator–prey model we introduce a planar, noninvertible map composed of a rigid rotation over an angle φ and a quadratic map depending on a parameter a. We study the dynamics of this map with a particular emphasis on the transitions from orderly to chaotic dynamics. For $a = 3$ a stable fixed point bifurcates through a Hopf–Neĭmark–Sacker bifurcation which gives rise to the alternation of periodic and quasi-periodic dynamics organized by Arnold tongues in the (φ, a)-plane. Inside a tongue a periodic attractor typically either undergoes a period doubling cascade, which leads to chaotic dynamics, or a Hopf–Neĭmark–Sacker bifurcation, which leads in turn to a new family of Arnold tongues. Numerical evidence suggests the existence of strange attractors with both one and two positive Lyapunov exponents. The former attractors are conjectured to be Hénon-like, i.e., they are formed by the closure of the unstable manifold of a periodic point of saddle type. The folded nature of such attractors is the novel feature of this paper.

Keywords: Endomorphisms; Bifurcations; (quasi-)periodic dynamics; Chaotic dynamics

1. Introduction

In this paper we study the dynamics of a planar endomorphism which is composed of a fold and a rigid rotation. We present both analytic computations and numerical experiments in which educated guesses are inspired by the available theory.

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1.1. Motivation

A central problem in the mathematical theory of dynamical systems is to determine the geometric structure of strange attractors and the bifurcations leading to their formation. A classical example for which rigorous results are available is the Hénon-map [19]. Benedicks and Carleson [3,4] proved that there exists a set of positive measure in the parameter plane for which the Hénon map has a strange attractor which coincides with the closure of the unstable manifold of a saddle fixed point.

Broer et al. [8] studied the dynamics of the so-called fattened Arnold map, which can be regarded as a perturbation of the Arnold family of circle maps. This map can be regarded as a simplified global model for the return map of a dissipative diffeomorphism near homoclinic bifurcation, which implies that its dynamics has a universal character in the context of 2-dimensional maps. This dynamics involves periodicity, quasiperiodicity and chaos, between which there are various bifurcations, and in this respect the map can be compared with examples like the Hénon map and the standard map.

Detailed studies of the dynamics of 3-dimensional diffeomorphisms, in particular near a Hopf-saddle–node bifurcation, can be found in [11,12,29]. These studies were mainly inspired by results obtained for the Poincaré map of the periodically driven Lorenz-84 atmospheric model [9]. In the latter map so-called quasi-periodic Hénon-like attractors, which are conjectured to coincide with the unstable manifold of a hyperbolic invariant circle of saddle-type, have been detected [10]. The existence of such attractors has been rigorously proved for a map on the solid torus [13].

Research on the dynamics of endomorphisms, i.e. maps which are not necessarily invertible, goes back to at least the works of Gumowski and Mira [17,18,23] who studied the role of critical lines in the formation of basin boundaries and their bifurcations. Since the 1990s the interest for 2-dimensional endomorphisms has increased tremendously. For a detailed account the reader is referred to the textbook of Mira et al. [25] and the references therein.

In this paper we study the dynamics of a particular planar endomorphism which is defined by the composition of a quadratic map and a rigid rotation. We are motivated by a predator–prey model and a discretization of the Lorenz-63 model in a limiting case, which both feature the combined effect of rotating and folding the phase plane. We study the attractors of our model map and their bifurcations. In particular, we study whether Hénon-like strange attractors can occur.

1.2. Object of study

In order to introduce our model map we first discuss two particular planar endomorphisms which rotate and fold the plane.

Predator–prey models. Consider the following planar map

$$P(x, y) = (ax(1 - x - y), bxy). \tag{1}$$

The map P is a simplification of the predator–prey model studied in [2]; also see [25] and references therein. This map has a fixed point $(\frac{1}{b}, 1 - \frac{1}{a} - \frac{1}{b})$, which has complex eigenvalues for the parameters $b > (a + \sqrt{a})/(2a - 2)$. Hence, the map P rotates points near this fixed point. In the half plane defined by the inequality $ab - 4bx - 4ay > 0$ the map P has two preimages.

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