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A generalization of the Erdös–Surányi problem

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Received 1 July 2016; received in revised form 20 July 2016; accepted 2 August 2016

Communicated by R. Tijdeman

Abstract

Erdös–Surányi and Prielipp suggested to study the following problem: For any integers $k > 0$ and n, are there an integer N and a map $\epsilon : \{1, \dots, N\} \to \{-1, 1\}$ such that

$$n = \sum_{j=1}^{N} \epsilon(j) j^k \ ? \tag{0.1}$$

Mitek and Bleicher independently solved this problem affirmatively.

In this paper we consider the case that for some positive odd integer L the numbers $\epsilon(j)$ are L-th roots of unity. We show that the answer to the corresponding question is negative if and only if L is a prime power.

Keywords: Theorem of Erdös and Surányi; The set of roots of unity; Prielipp's problem; Representation of integers; Signed sums

1. Introduction

Erdös–Surányi [7] and Prielipp [3] suggested to study the following problem: For any integers $k > 0$ and n, are there an integer N and a map $\epsilon : \{1, \dots, N\} \to \{-1, 1\}$ such that

$$n = \sum_{j=1}^{N} \epsilon(j) j^k \ ? \tag{1.1}$$

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http://dx.doi.org/10.1016/j.indag.2016.08.002

Please cite this article in press as: E. Miyanohara, A generalization of the Erdös–Surányi problem, Indagationes Mathematicae (2016), http://dx.doi.org/10.1016/j.indag.2016.08.002

Mitek [8] and Bleicher [3] independently solved this problem affirmatively. Later many people investigated analogies and generalizations of this problem (see [1,2,5,6,9]). Some researchers replaced the function ϵ by another function [4,5].

We study the case that the values of ϵ are Lth roots of unity, where L is a positive integer. Since, by [3,8], we already know that the answer is positive if L is even, we restrict our attention to odd L. Let U be the set of Lth roots of unity. Then we consider the following problem (cf. [4]). For any integers $k > 0$ and n, are there an integer N and a map $\epsilon : \{1, \dots, N\} \to U$ such that

$$n = \sum_{j=1}^{N} \epsilon(j) j^k? \tag{1.2}$$

We prove the following result.

Theorem 1.1. *Let L be a positive odd integer with $L \geq 2$ which is not a prime power and let U be the set of Lth roots of unity.*

Then for any integers $k > 0$ and n, there are an integer N and a map $\epsilon : \{1, \dots, N\} \to U$ such that

$$n = \sum_{j=1}^{N} \epsilon(j) j^k. \tag{1.3}$$

The following result shows that the statement of Theorem 1.1 is valid if L is an odd prime power p^m and k is a multiple of $p - 1$.

Theorem 1.2. *Let p be an odd prime number, m be a positive integer and let U be the set of p^mth roots of unity. Then for any integers $k > 0$ with $p - 1 \mid k$ and n, there are an integer N and a map $\epsilon : \{1, \dots, N\} \to U$ such that*

$$n = \sum_{j=1}^{N} \epsilon(j) j^k. \tag{1.4}$$

Moreover the following result shows that the statement of Theorem 1.1 is not valid if L is an odd prime power p^m and k is not a multiple of $p - 1$.

Theorem 1.3. *Let p be an odd prime number, m be a positive integer and let U be the set of p^mth roots of unity. Then for any integer $k > 0$ with $p - 1 \nmid k$, there are infinitely many integers n such that n cannot be represented as*

$$n = \sum_{j=1}^{N} \epsilon(j) j^k, \tag{1.5}$$

where N is a positive integer and $\epsilon : \{1, \dots, N\} \to U$.

Remark 1.4. Theorem 1.3 contradicts Theorem 5.3 in [4]. The proof of Proposition 4.2 of [4] contains a serious error. Let μ_K, $\mathcal{R}$, $\varepsilon D_m[f](l)$ and $\varepsilon \overline{D_m}[f](l)$ be defined as in [4]. Since μ_K need not contain -1, it may be that $\varepsilon D_m[f](l)$ and $\varepsilon \overline{D_m}[f](l)$ are not contained in $\mathcal{R}$.

2. Proof of Theorem 1.1

First we generalize Lemma 3 in [3] as follows:

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