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On uniform approximation to successive powers of a real number

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Abstract

We establish new inequalities involving classical exponents of Diophantine approximation. This allows for improving on the work of H. Davenport, W.M. Schmidt and M. Laurent concerning the maximum value of the exponent $\widehat{\lambda}_n(\zeta)$ among all real transcendental ζ. In particular we refine the estimation $\widehat{\lambda}_n(\zeta) \leq \lceil n/2 \rceil^{-1}$ due to M. Laurent by $\widehat{\lambda}_n(\zeta) \leq \widehat{w}_{\lceil n/2 \rceil}(\zeta)^{-1}$ for all $n \geq 1$, and for even n we replace the bound $2/n$ for $\widehat{\lambda}_n(\zeta)$ first found by Davenport and Schmidt by roughly $\frac{2}{n} - \frac{4}{n^3}$, which provides the currently best known bound when $n \geq 6$.

Keywords: Simultaneous approximation; Geometry of numbers; Successive minima

1. Introduction

1.1. Exponents of approximation

We start with the definition of classical exponents of Diophantine approximation. Let ζ be a real transcendental number and $n \geq 1$ be an integer. For $1 \leq j \leq n+1$ we define the exponents of simultaneous approximation $\lambda_{n,j}(\zeta)$ as the supremum of $\eta \in \mathbb{R}$ such that the system

$$|x| \leq X, \qquad \max_{1 \leq i \leq n} |\zeta^i x - y_i| \leq X^{-\eta}, \tag{1}$$

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has (at least) j linearly independent solutions $(x, y_1, y_2, \dots, y_n) \in \mathbb{Z}^{n+1}$ for arbitrarily large values of X. Moreover, let $\widehat{\lambda}_{n,j}(\zeta)$ be the supremum of η such that (1) has (at least) j linearly independent solutions for all $X \geq X_0$. In case of $j = 1$ we also only write $\lambda_n(\zeta)$ and $\widehat{\lambda}_n(\zeta)$ respectively, which coincide with the classical exponents of approximation defined by Bugeaud and Laurent [4]. By Dirichlet's Theorem, for all $n \geq 1$ and transcendental real ζ, these exponents are bounded below by

$$\lambda_n(\zeta) \geq \widehat{\lambda}_n(\zeta) \geq \frac{1}{n}. \tag{2}$$

Moreover it is clear from the definition that

$$\lambda_1(\zeta) \geq \lambda_2(\zeta) \geq \cdots, \qquad \widehat{\lambda}_1(\zeta) \geq \widehat{\lambda}_2(\zeta) \geq \cdots. \tag{3}$$

Similarly, for $1 \leq j \leq n+1$ let $w_{n,j}(\zeta)$ and $\widehat{w}_{n,j}(\zeta)$ be the supremum of $\eta \in \mathbb{R}$ such that the system

$$H(P) \leq X, \qquad 0 < |P(\zeta)| \leq X^{-\eta}, \tag{4}$$

has (at least) j linearly independent polynomial solutions $P(T) = a_n T^n + a_{n-1}T^{n-1} + \cdots + a_0$ of degree at most n with integers a_j for arbitrarily large X and all large X, respectively. Here $H(P) = \max_{0\leq j\leq n} |a_j|$ is the height of P as usual. Again for $j = 1$ we also write $w_n(\zeta)$ and $\widehat{w}_n(\zeta)$, which coincide with classical exponents. Dirichlet's Theorem implies the estimates

$$w_n(\zeta) \geq \widehat{w}_n(\zeta) \geq n. \tag{5}$$

Moreover it is obvious that the exponents satisfy the relations

$$w_1(\zeta) \leq w_2(\zeta) \leq \cdots, \qquad \widehat{w}_1(\zeta) \leq \widehat{w}_2(\zeta) \leq \cdots. \tag{6}$$

We point out that identities due to Mahler can be stated in the form

$$\lambda_{n,j}(\zeta) = \frac{1}{\widehat{w}_{n,n+2-j}(\zeta)}, \qquad 1 \leq j \leq n+1, \tag{7}$$

$$w_{n,j}(\zeta) = \frac{1}{\widehat{\lambda}_{n,n+2-j}(\zeta)}, \qquad 1 \leq j \leq n+1, \tag{8}$$

as carried out in [17]. In particular (8) will be of major importance for us, since it is essential in the proof of our main result Theorem 2.1. The remaining results we now state below again concern the best approximation constants, i.e. $j = 1$ in the above definitions. For the remainder of this section all results are implicitly understood to be valid for all $n \geq 1$ and all real transcendental ζ, unless stated otherwise. We remark that some of the results can be extended to the case of vectors $\underline{\zeta} \in \mathbb{R}^n$ which are $\mathbb{Q}$-linearly independent together with $\{1\}$. Khintchine's transference inequalities [11] connect the exponents of best approximation with the polynomial approximation constants via

$$\frac{w_n(\zeta)}{(n-1)w_n(\zeta) + n} \leq \lambda_n(\zeta) \leq \frac{w_n(\zeta) - n + 1}{n}. \tag{9}$$

German [10] established inequalities involving the uniform constants given by

$$\frac{\widehat{w}_n(\zeta) - 1}{(n-1)\widehat{w}_n(\zeta)} \leq \widehat{\lambda}_n(\zeta) \leq \frac{\widehat{w}_n(\zeta) - n + 1}{\widehat{w}_n(\zeta)}. \tag{10}$$

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