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Q1 # A local regularity result for Neumann parabolic problems with nonsmooth data

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Abstract

In this work we analyze the relations between two different concepts of solution of the Neumann problem for a second order parabolic equation: the usual notions of weak solution and those of transposition solution, which allow well-posedness of problems with measure data. We give a regularity result for the transposition solution and we prove that, under smoothness assumptions for the principal part of the operator, the local regularity of the transposition solution is the same as that of the usual weak solution. As an interesting particular case, we present a rigorous proof of local continuity of the solution for a convection–diffusion problem with pointwise source term.

Keywords: Weak solution; Transposition solution; Convection–diffusion; Measure data; Neumann boundary condition Q3

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1. Introduction

1.1. Preliminaries

Let Ω be a bounded domain of $\mathbb{R}^n$ with Lipschitz boundary Γ. For $T > 0$, we denote $Q_T = \Omega \times (0, T)$, and $\Sigma_T = \Gamma \times (0, T)$.

We consider the general second order parabolic problem with Neumann boundary condition:

$$\left.\begin{aligned} \frac{\partial y}{\partial t} + Ly = f \quad & \text{in } Q_T, \\ \frac{\partial y}{\partial \nu_L} = g \quad & \text{on } \Sigma_T, \\ y(x,0) = y_0 \quad & \text{in } \Omega, \end{aligned}\right\} \tag{1}$$

where

$$Ly = -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left\{\sum_{j=1}^n a_{ij}(x,t)\frac{\partial y}{\partial x_j} + a_i(x,t)y\right\} + \sum_{i=1}^n b_i(x,t)\frac{\partial y}{\partial x_i} + c(x,t)y,$$

$$\frac{\partial y}{\partial \nu_L} = \sum_{i=1}^n \left\{\sum_{j=1}^n a_{ij}(x,t)\frac{\partial y}{\partial x_j} + a_i(x,t)y\right\}\nu_i(x),$$

for $\vec{\nu}(x) = (\nu_1(x), \dots, \nu_n(x))$ the outward unit normal vector to Γ in the point $x \in \Gamma$. We assume the following standard hypotheses on the coefficients of the operator L:

- $a_{ij} \in L^\infty(Q_T),\ 1 \le i, j \le n,$
- $a_i, b_i \in L^\infty(Q_T)\ 1 \le i \le n,$
- $c \in L^\infty(Q_T),$
- $\exists\, \alpha > 0 \,/\, \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x,t)\xi_i\xi_j \ge \alpha|\xi|^2,\ \forall \xi \in \mathbb{R}^n,$ a.e. $(x,t) \in Q_T.$

Finally we introduce the associated family of bilinear forms

$$a(t; .,.) : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}$$

defined by:

$$\begin{aligned} a(t; w, v) = {} & \int_\Omega \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x,t)\frac{\partial w}{\partial x_j}\frac{\partial v}{\partial x_i}\,dx + \int_\Omega \sum_{i=1}^n a_i(x,t)w\frac{\partial v}{\partial x_i}\,dx \\ & + \int_\Omega \sum_{i=1}^n b_i(x,t)\frac{\partial w}{\partial x_i}v\,dx + \int_\Omega c(x,t)wv\,dx, \end{aligned} \tag{2}$$

and the family of operators $A(t) \in \mathcal{L}(H^1(\Omega), [H^1(\Omega)]')$ defined by:

$$\langle A(t)w, v\rangle = a(t; w, v), \quad \forall\, w, v \in H^1(\Omega), \text{ a.e. } t \in (0, T).$$

We shall also use the notation $a(t; w, v)$ with the same meaning as in Eq. (2) when $w \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$ with $p \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

All along the mathematical literature several concepts of weak solution for second order parabolic equations have been given (see, for instance, Ladyzenskaja et al. [19],

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