Available online at www.sciencedirect.com

ScienceDirect

Indagationes Mathematicae 28 (2017) 556–565

indagationes mathematicae

www.elsevier.com/locate/indag

On a relation between spectral theory of lens spaces and Ehrhart theory

H. Mohades, B. Honari*

Department of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran

Received 6 October 2015; received in revised form 4 January 2017; accepted 10 January 2017

Communicated by E.P. van den Ban

Abstract

In this article Ehrhart quasi-polynomials of simplices are employed to determine isospectral lens spaces in terms of a finite set of numbers. Using the natural lattice associated with a lens space the associated toric variety of a lens space is introduced. It is proved that if two lens spaces are isospectral then the dimension of global sections of powers of a natural line bundle on these two toric varieties are equal and they have the same general intersection number. Also, harmonic polynomial representation of the group $SO(n)$ is used to provide a more elementary proof for a theorem of Lauret, Miatello and Rossetti on isospectrality of lens spaces.

Keywords: Lens space; Laplace–Beltrami operator; Isospectrality; Harmonic polynomial; Ehrhart quasi-polynomial; Rational convex polytope; Toric variety; Divisor; Line bundle

1. Introduction

The Laplace–Beltrami operator is a natural second order elliptic operator on a Riemannian manifold defined as **div** ∘ **grad**. It is well known that on a closed manifold, this operator has discrete positive eigenvalues with finite multiplicities [4]. Two Riemannian manifolds are isospectral if their Laplace–Beltrami operators have the same spectrum, considering multiplicities.

* Corresponding author.
E-mail addresses: hoseinmohades@aut.ac.ir (H. Mohades), honari@aut.ac.ir (B. Honari).

http://dx.doi.org/10.1016/j.indag.2017.01.003

A fundamental question by Mark Kac asks whether it is possible to find two nonisometric isospectral manifolds. The first negative answer to this question was provided by Milnor's 16 dimensional tori that are geometric realizations of self-dual lattices with the same theta functions [23]. Nowadays there are so many methods to construct such manifolds, including Vigneras' arithmetic method which benefits from quaternion algebras [28], Sunada's triple method and its generalizations based on free action of a triple (G, H_1, H_2) consisting of a group G and two of its almost conjugate subgroups that provide an equivalent linear representation for regular representations which are not permutationally equivalent [27,11,10,9], Gordon's torus bundle method [3], etc. There are some examples of isospectral manifolds that cannot be derived from the above general methods including isospectral lens spaces introduced by Ikeda [15,14,13,16,17].

Necessity of calculation of analytic torsions led Ray to present explicitly the spectrum of lens spaces. That was actually a continuation of works of Calabi, Gallot and Meyer on spectrum of spheres [24]. Ikeda used representation theory to find the spectrum of lens spaces. He also used his subtle method to study the spectrum of Hodge–Laplace operator on differential forms [15,17]. Representation theory methods have been applied by Ikeda, Gordon, Gornet, Lauret, McGowan, Miatello, Rossetti, and others to construct examples of spherical orbifolds and space forms that clarify differences between many types of isospectrality regarding the spectrum of elliptic differential operators on these spaces [8,12,9,20,21]. Recently Lauret, Miatello, and Rossetti extended representation theoretic methods to find conditions on the natural lattices Γ associated with isospectral lens spaces [20]. These conditions completely determine isospectral lens spaces. Lauret also has recently applied the Ehrhart theory to derive a geometric characterization of isospectral orbifolds with cyclic fundamental group [19]. On the other hand, Ehrhart proved that the number of points of the lattice $\mathbb{Z}^n$ which lie inside an integral multiple of a rational polygon can be obtained by a rational function [6]. In the case of a rational simplex, Macdonald, Stanley and others gave an explicit formula for this rational function, so they provided an explicit formula for the number of lattice points inside simple polytopes [2,22,25]. Motivated by the theorem of Lauret, Miatello, and Rossetti, we naturally associate a set of simplices with a lens space. Using these simplices, we introduce the Ehrhart polynomial and associated toric variety of a lens space and we consider the effect of isospectrality of lens spaces on these objects. For the sake of completeness, a direct proof of Lauret, Miatello, and Rosetti's theorem is presented by using the theory of harmonic polynomials as a realization for representation theory of $\mathrm{SO}(n)$.

Here an overview of the paper is given. In Section 2, preliminaries on lens spaces are presented. Ehrhart polynomial is introduced in Section 3. In Section 4 a proof of Lauret, Miatello, and Rosetti's theorem is presented. In Section 5 methods of previous sections are used to find conditions for isospectrality of lens spaces. Also the toric variety associated with a lens space is introduced and it is proved that if two lens spaces are isospectral then the dimension of global sections of powers of a natural line bundle on these two toric varieties are equal and they have the same general intersection number.

2. Preliminaries on lens spaces

2.1. Harmonic homogeneous polynomials

A multivariate polynomial P on n variables with coefficients in $\mathbb{C}$ is homogeneous of degree m if and only if $P(\lambda x_1, \ldots, \lambda x_n) = \lambda^m P(x_1, \ldots, x_n)$ for every $\lambda \in \mathbb{C}$. The dimension of the complex vector space generated by these functions is equal to the number of different monomials