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Hyperbolic imbeddedness and tautness modulo a closed subset

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Abstract

In the present paper, we study some properties of the Kobayashi metric on unbounded convex domains and on Hartogs domains defined by $\Omega_\varphi(\mathbb{C}^m) = \{(z, w) \in \mathbb{C}^m \times \mathbb{C} : |w| < e^{-\varphi(z)}\}$, $m \in \mathbb{N}^*$, where φ is a smooth strictly plurisubharmonic function such that $\varphi(z) \to +\infty$ as $|z| \to \infty$.

Keywords: Complex manifold; Kobayashi hyperbolicity; Hyperbolic imbeddedness; Hartogs domain; Tautness

1. Introduction

Introduced in 1967 by S. Kobayashi, the Kobayashi pseudometric, or pseudodistance, is an important biholomorphically invariant on complex manifolds. It can be viewed as a natural generalization of the Poincaré distance of the unit disc in $\mathbb{C}$ and has been used for the study of holomorphic maps and function spaces in several complex variables. In case the pseudometric is a distance (the manifold is said to be Kobayashi hyperbolic), then the manifold has important geometric properties; for instance the automorphism group of the manifold is a real Lie group. Proving that a complex manifold is Kobayashi hyperbolic is an important and difficult question. For a compact manifold, this is related to the Kobayashi conjecture (hyperbolicity of high degree hypersurface) or the Green–Griffiths–Lang conjecture.

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In this paper, we consider open complex manifolds and more precisely domains in the complex euclidean space $\mathbb{C}^n$. If a bounded domain is always hyperbolic, the situation is subtle and difficult for generic unbounded domains that appear naturally in the geometric study of complex manifolds. Almost no result is known on the behavior of invariant metrics on such domains, see for instance [2] where the authors give a complete characterization of complete hyperbolic convex domains, and [3,1] where the authors study the behavior of the Bergman metric on unbounded domains [1].

In a recent paper [6] T. Harz, N. Shcherbina, G. Tomassini studied obstructions, called the core of the domain, for a strongly pseudoconvex unbounded domain to admit a global strictly plurisubharmonic function.

We prove here that a strictly pseudoconvex domain Ω with a smooth boundary is hyperbolic modulo $\mathfrak{c}(\Omega)$ and as an application, we consider the Hartogs domain given by

$$\Omega_\varphi(\mathbb{C}^m) = \left\{(z, w) \in \mathbb{C}^m \times \mathbb{C} : |w| < e^{-\varphi(z)}\right\}, \quad m \in \mathbb{N}^*$$

where φ is a smooth strictly plurisubharmonic function such that $\varphi(z) \to +\infty$, as $|z| \to \infty$.

We show that $\mathfrak{c}(\Omega_\varphi(\mathbb{C}^m)) = \mathbb{C}^m \times \{0\}$, hence, $\Omega_\varphi(\mathbb{C}^m)$ is hyperbolic modulo $S = \mathbb{C}^m \times \{0\}$. Next we show that $\Omega_\varphi(\mathbb{C}^m)$ is taut modulo S and $\Omega_\varphi(\mathbb{C}^m) \setminus S$ is complete hyperbolic. Then, we show that a convex domain in $\mathbb{C}^n$ (probably unbounded) is taut modulo a closed subset if and only if it is taut.

In a series of papers, different authors studied metric properties of Hartogs type domains and hyperbolic imbeddedness of complex spaces (see, for instance [14,11–13,4]). We use similar techniques to study the hyperbolic imbeddedness (modulo a closed subset) of Hartogs type domains.

The paper is organized as follows. In Section 2 we recall some notations and definitions needed to the comprehension of the paper. Also, some useful results are established and will be used later.

In Section 3, we study unbounded Hartogs domain and we also prove that a convex domain is taut modulo a closed subset if and only if it is taut.

Finally, we study hyperbolic imbeddedness of Hartogs domains modulo the core (see Definition 3.1) in Section 4.

2. Preliminaries

In this paper, we will use the following notations:

- For $z \in \mathbb{C}^n$, $|z|$ denotes the standard Euclidean norm of z.
- $\Delta = \{z \in \mathbb{C}, |z| < 1\}$ (resp. $\Delta^* = \{z \in \mathbb{C}, 0 < |z| < 1\}$) the unit disc in $\mathbb{C}$ and $\Delta_R = \{z \in \mathbb{C}, |z| < R\}$ (resp. $\Delta_R^* = \{z \in \mathbb{C}, 0 < |z| < R\}$),
- For $z_0 \in \mathbb{C}$ and $r > 0$, $\Delta(z_0, r) = \{\zeta \in \mathbb{C} : |\zeta - z_0| < r\}$.

Definition 2.1. Given a complex manifold X, the Kobayashi infinitesimal pseudometric is the pseudo-Finsler metric

$$K_X(x, v) = \inf\left\{\frac{1}{r} : f \in \mathrm{Hol}(\Delta_r, X), f(0) = x, f'(0) = v\right\}.$$

Given a piecewise smooth $\mathcal{C}^1$ curve $\gamma : [0, 1] \longrightarrow X$, the length of γ is

$$l_X(\gamma) = \int_0^1 K(\gamma(t), \gamma'(t))dt.$$