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Markov–Kakutani's theorem for best proximity pairs in Hadamard spaces

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Abstract

In the current paper, we consider two classes of noncyclic mappings, called quasi-noncyclic relatively nonexpansive and noncyclic relatively u-continuous, and survey the existence of best proximity pairs as well as the structure of best proximity pair sets for these classes of mappings in Busemann convex spaces. We also study the existence of a common best proximity pair for families if noncyclic mappings in Hadamard spaces. In this way, we obtain a generalization of Markov–Kakutani's fixed point theorem in Hadamard spaces.

Keywords: Best proximity pair; Geodesic metric space; Noncyclic mapping; Proximal normal structure

1. Introduction

Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be *noncyclic* provided that $T(A) \subseteq A$ and $T(B) \subseteq B$. Moreover, T is said to be a *cyclic mapping* if $T(A) \subseteq B$ and $T(B) \subseteq A$. A point $(p, q) \in A \times B$ is said to be a *best proximity pair* for the noncyclic mapping T provided that p and q are two fixed points of T and the distance

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between two points p and q estimates the distance between two sets A and B. In other words, $(p, q) \in A \times B$ is a best proximity pair for the noncyclic mapping T if

$$p = Tp, \qquad q = Tq \quad \text{and} \quad d(p, q) = \text{dist}(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}.$$

In the case that T is a cyclic mapping, a point $p \in A \cup B$ is called a *best proximity point* whenever $d(p, Tp) = \text{dist}(A, B)$.

The mapping $T : A \cup B \to A \cup B$ is said to be *relatively nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $(x, y) \in A \times B$. It is interesting to note that noncyclic relatively nonexpansive mappings may not be continuous, necessarily. In a special case, if $A = B$, then T is said to be a nonexpansive mapping.

In [8] the existence results of best proximity pairs (best proximity points) were studied for noncyclic (cyclic) relatively nonexpansive mappings in a Banach space X by using a geometric property on a nonempty and convex pair of subsets of X, called *proximal normal structure*, (Theorems 2.1, 2.2 of [8]). Thereby, an interesting extension of *Kirk's fixed point theorem* [17] was concluded.

Recently, another class of noncyclic mappings was introduced in [19] as below (see also [9,16] for more details).

Definition 1.1 (*[9,19]*)**.** Let A and B be two nonempty subsets of a Banach space X. A mapping $T : A \cup B \to A \cup B$ is said to be a noncyclic (cyclic) relatively u-continuous mapping if T is noncyclic (cyclic) on $A \cup B$ and satisfies the following condition:

$$\forall \varepsilon > 0, \ \exists \delta > 0; \quad \text{if } \|x - y\|^* < \delta \text{ then } \|Tx - Ty\|^* < \varepsilon,$$

for all $(x, y) \in A \times B$, where $\|x - y\|^* := \|x - y\| - \text{dist}(A, B)$.

We mention that the class of noncyclic (cyclic) relatively u-continuous mappings contains the class of noncyclic (cyclic) relatively nonexpansive mappings as a subclass.

Next existence theorem was proved in [9,19].

Theorem 1.2 (*See Theorem 3.1 of [9] and Theorem 14 of [19]*)**.** *Let A and B be two nonempty, compact and convex subsets of a strictly convex Banach space X and let $T : A \cup B \to A \cup B$ be a noncyclic (cyclic) relatively u-continuous mapping. Then T has a best proximity pair (best proximity point).*

In this article, we discuss on the existence of best proximity pairs for another class of noncyclic mappings, called *quasi-noncyclic relatively nonexpansive*, which is a different class w.r.t. the class of noncyclic relatively nonexpansive mappings. Then we study the existence of a common best proximity pair for two commuting noncyclic relatively u-continuous and quasi-noncyclic relatively nonexpansive mappings in Hadamard spaces. Finally, we obtain an extension version of Markov–Kakutani's theorem for best proximity pairs.

2. Preliminaries

A metric space (X, d) is said to be a *(uniquely) geodesic space* if every two points x and y of X are joined by a *(unique) geodesic*, i.e., a map $c : [0, l] \subseteq \mathbb{R} \to X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. A subset A of a geodesic space X is said to be *convex* if the image of any geodesic that joins each pair of points x and y of A (geodesic segment $[x, y]$) is contained in A. A point z in X belongs to a geodesic segment $[x, y]$ if and only if

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