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Indagationes Mathematicae 28 (2017) 694–710

indagationes mathematicae

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Wiener–Hopf indices of unitary functions on the unit circle in terms of realizations and related results on Toeplitz operators

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Received 11 November 2016; received in revised form 18 January 2017; accepted 24 March 2017

Communicated by: H.J. Woerdeman

Abstract

We provide new formulas for the Wiener–Hopf factorization indices of a rational matrix function R which has neither poles nor zeros on the unit circle. In addition, we recover recent results on the Fredholm characteristics of the Toeplitz operator with symbol R via the method of matricial coupling. Furthermore, we present an alternative formula for the index in terms of the Fourier coefficients of R.

Keywords: Wiener–Hopf indices; Toeplitz operators

1. Introduction and preliminaries

The main goal of this paper is to provide formulas for the Wiener–Hopf indices of a rational $m \times m$ matrix function R which has no poles and zeros on the unit circle and has unitary values

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http://dx.doi.org/10.1016/j.indag.2017.03.001
0019-3577/

there. Moreover, we shall also provide an alternative formula for the index of the associated Toeplitz operator in terms of the Fourier coefficients of R.

We start with some notation and terminology. Recall that a *right Wiener–Hopf factorization* with respect to the unit circle (see, for instance, [3,5,6]) is a factorization

$$R(z) = W_-(z)D(z)W_+(z),$$

where the factors W_- and W_+ are rational $m \times m$ matrix functions such that

(i) W_- has no poles and zeros outside the open unit disc including infinity,
(ii) W_+ has no poles and zeros on the closed unit disc,

and where the middle term D is a diagonal matrix

$$D(z) = \operatorname{diag}(z^{-\alpha_1}, \ldots, z^{-\alpha_s}) \oplus I_k \oplus \operatorname{diag}(z^{\omega_t}, \ldots, z^{\omega_1}),$$

where $-\alpha_1 \le \cdots \le -\alpha_s < 0$ and $0 < \omega_t \le \cdots \le \omega_1$ are integers, and $m = s + k + t$. The numbers α_j and ω_j are called the *right Wiener–Hopf indices*. Introducing

$$\kappa_j = \begin{cases} -\alpha_j, & j = 1, \ldots, s, \\ 0, & j = s+1, \ldots, m-t, \\ \omega_{m-j+1}, & j = m-t+1, \ldots, m, \end{cases}$$

the middle term $D(z)$ can be written as $D(z) = \operatorname{diag}(z^{\kappa_1}, \ldots, z^{\kappa_m})$. Reversing the roles of W_- and W_+ one obtains the definition of a *left Wiener–Hopf factorization* and *left Wiener–Hopf indices*. If $\kappa_j = 0$ for all j, the factorization is called a right canonical factorization.

There is an intimate relation between the Wiener–Hopf indices and the Toeplitz operator T_R acting on $\ell^2_+(\mathbb{C}^m)$ with defining function R, as follows:

$$n(T_R) := \dim \operatorname{Ker} T_R = \sum_{j=1}^{s} \alpha_j = \sum_{\kappa_j \le 0} -\kappa_j,$$

$$d(T_R) := \operatorname{codim} \operatorname{Im} T_R = \sum_{j=1}^{t} \omega_j = \sum_{\kappa_j \ge 0} \kappa_j,$$

$$\operatorname{ind} T_R := n(T_R) - d(T_R) = \sum_{j=1}^{s} \alpha_j - \sum_{j=1}^{t} \omega_j,$$

Theorem VIII.5.1 in [11], also Theorem 2.18 in [15], while for an analogous result on singular integral operators we refer to [6,15]. Even more precisely, introduce $R_k(z) = z^{-k}R(z)$, then we have, see [13], Section XXIV.4, page 590,

$$\#\{j \mid \kappa_j = k\} = \left(d(T_{R_{k+1}}) - d(T_{R_k})\right) - \left(d(T_{R_k}) - d(T_{R_{k-1}})\right).$$

The problem we are interested in is to find the Wiener–Hopf indices explicitly in terms of a realization of the symbol R. For the case of matrix polynomials this was carried out in [16] and [20]. The special case of rational matrix functions which are analytic and invertible at infinity was done in detail in [10], and in [2,3], see also [5] Theorem 7.8, in terms of proper realizations, and in [14] with a different approach. However, the restriction on the proper realization of regularity at infinity is severe: indeed, the diagonal term $D(z) = \operatorname{diag}(z^{\kappa_1}, \ldots, z^{\kappa_m})$ does not fall into the class of functions that are regular at infinity. A description of the Wiener–Hopf indices in terms of observability and controllability indices of certain pairs of matrices coming