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On weak model sets of extremal density

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Abstract

The theory of regular model sets is highly developed, but does not cover examples such as the visible lattice points, the kth power-free integers, or related systems. They belong to the class of weak model sets, where the window may have a boundary of positive measure, or even consists of boundary only. The latter phenomena are related to the topological entropy of the corresponding dynamical system and to various other unusual properties. Under a rather natural extremality assumption on the density of the weak model set, we establish its pure point diffraction nature. We derive an explicit formula that can be seen as the generalisation of the case of regular model sets. Furthermore, the corresponding natural patch frequency measure is shown to be ergodic. Since weak model sets of extremal density are generic for this measure, one obtains that the dynamical spectrum of the hull is pure point as well.

1. Introduction

The theory of regular model sets, which are also known as cut and project sets with sufficiently nice windows, is well established; see [1] and references therein for general background. The vertex set of the rhombic Penrose tiling, and of many other related tilings, are widely known examples of regular model sets [14,1]. One cornerstone of this class is the pure pointedness of the diffraction measure [21,37,8]. Equivalently, this means that the dynamical spectrum of the uniquely ergodic hull defined by the model set is pure point as well; compare [25,5,27]. The regularity of the window is vital to the existing proofs such as that in [37], and also enters the characterisation of regular model sets via dynamical systems [6].

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For quite some time, systems such as the visible lattice points or the kth power-free integers have been known to be pure point diffractive as well [9]. These point sets can also be described as model sets, but here the windows are no longer regular. In fact, for each of these examples, the window consists of boundary only, which has positive measure, and many other properties of regular model sets are lost, too. In particular, there are many invariant probability measures on the orbit closure (or hull) of the point set under the translation action of the lattice. Yet, as explicit recent progress has shown, the natural cluster (or patch) frequency measure of this hull is ergodic and the visible points are generic for this measure [3]. Consequently, the dynamical spectrum is still pure point, by an application of the general equivalence theorem [5]. Since this example is one out of a large class with similar properties, it is natural to ask for a general approach that includes all of them. Such a class is provided by *weak model sets*, where one allows more general windows. This name was coined by Moody [31,32], see also [1, Rem. 7.4], while this class seems to have first been looked at by Schreiber [38].

It is the purpose of this paper to derive some key results for weak model sets. To this end, we begin with the visible lattice points as a motivating example. Then, we start from the general setting of model sets for a general cut and project scheme $(G, H, \mathcal{L})$, see Eq. (2) for a definition, but investigate the diffraction properties for the larger class of windows indicated above. It turns out that this is indeed possible under one fairly natural assumption, namely that of maximal or minimal density with respect to a given van Hove averaging sequence in the group G. This assumption guarantees pure point diffractivity (Theorems 7 and 9).

In a second step, we analyse the ergodicity of the cluster frequency measure for the very van Hove sequence, which then implies the dynamical properties of the hull we are after. In particular, we establish that weak model sets of extremal density have pure point dynamical spectrum, and calculate the latter. Finally, we apply our results to certain coprime lattice families, which encompasses the k-visible lattice points in d-space as well as other examples of arithmetic origin, such as k-free or (coprime) $\mathcal{B}$-free integers [34,3,17,10,24] and their generalisations to analogous systems in number fields [13,3,11]. This way, we demonstrate that and how the theory of weak model sets provides a natural framework for a unified treatment of such systems.

In parallel to our approach, Keller and Richard [23] have developed an alternative view on model sets via a systematic exploitation of the torus parametrisation for such systems; compare [2,19,37,6]. Their work includes weak model sets and provides an independent way to derive several of our key results. In this sense, the two approaches are complementary and, in conjunction, give a more complete picture of a larger class of model sets than understood previously, both concretely and structurally.

2. Preliminaries and background

Our general reference for background and notation is the recent monograph [1]. Here, we basically summarise some key concepts and their extensions in the generality we need them. Let G be a locally compact Abelian group (LCAG), and denote the space of translation bounded (and generally complex) Radon measures on G by $\mathcal{M}^{\infty}(G)$. Here and below, measures are viewed as linear functionals on the space $C_{\mathsf{c}}(G)$ of continuous functions with compact support, which is justified by the general Riesz–Markov theorem; see [12] for general background. In this setting, we use $\mu(g)$ and $\int_G g \, \mathrm{d}\mu$ for an integrable function g as well as $\mu(A) = \int_G 1_A \, \mathrm{d}\mu$ for a Borel set A exchangeably. For a measure μ, we define its twisted version $\widetilde{\mu}$ by $\widetilde{\mu}(g) = \overline{\mu(\widetilde{g})}$ for $g \in C_{\mathsf{c}}(G)$ as usual, where $\widetilde{g}(x) := \overline{g(-x)}$.

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