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Shrinking random β-transformation

Kan Jiang, Karma Dajani*

Department of Mathematics, Utrecht University, Fac Wiskunde en informatica and MRI, Budapestlaan 6, P.O. Box 80.000, 3508 TA Utrecht, The Netherlands

Abstract

For any $n \geq 3$, let $1 < \beta < 2$ be the largest positive real number satisfying the equation

$$\beta^n = \beta^{n-2} + \beta^{n-3} + \cdots + \beta + 1.$$

In this paper we define the shrinking random β-transformation K and investigate natural invariant measures for K, and the induced transformation of K on a special subset of the domain. We prove that both transformations have a unique measure of maximal entropy. However, the measure induced from the intrinsically ergodic measure for K is not the intrinsically ergodic measure for the induced system.

Keywords: Random β-transformation; Unique measure of maximal entropy; Invariant measure

1. Introduction

Let $\beta \in (1, 2)$ and $x \in \mathcal{A}_\beta = [0, (\beta - 1)^{-1}]$, we call a sequence $(a_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ a β-expansion of x if

$$x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}.$$

Renyi [11] introduced the greedy map, and showed that the greedy expansion $(a_i)_{i=1}^{\infty}$ of $x \in [0, 1)$ can be generated by defining $T(x) = \beta x \mod 1$ and letting $a_i = k$ whenever

* Corresponding author.
E-mail address: k.dajani1@uu.nl (K. Dajani).

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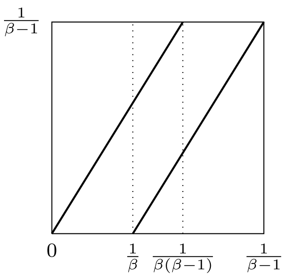

Fig. 1. The dynamical system for $\{T_0(x) = \beta x,\ T_1(x) = \beta x - 1\}$.

$T^{i-1}(x) \in [k\beta^{-1}, (k+1)\beta^{-1})$. Since then, many papers were dedicated to the dynamical properties of this map, see for example [12,5,8,10,4,9] and references therein. However, Renyi's greedy map is not the unique dynamical approach to generate β-expansions. In [6] (see also [3,4]) a new transformation was introduced, the random β-transformation, that generates all possible β-expansions, see Fig. 1. This transformation makes random choices between the maps $T_0(x) = \beta x$ and $T_1(x) = \beta x - 1$ whenever the orbit falls into $[\beta^{-1}, \beta^{-1}(\beta-1)^{-1}]$, which we refer to as the switch region.

Although, all possible β-expansions can be generated via the random β-transformation, nevertheless, for some practical problems one would want to make choices only on a subset of the switch region $[\beta^{-1}, \beta^{-1}(\beta-1)^{-1}]$, for instance, in A/D (analog-to-digit) conversion [7]. This motivates our study of the shrinking random β-transformation described below.

Let $1 < \beta < 2^{-1}(1+\sqrt{5})$, $\Omega = \{0,1\}^{\mathbb{N}}$, and $E = [0, (\beta-1)^{-1}]$. Set $a = (\beta^2-1)^{-1}$, $b = \beta(\beta^2-1)^{-1}$, i.e. $T_0(a) = b$, $T_1(b) = a$. The shrinking random β-transformation K is defined in the following way (see Fig. 2).

Definition 1.1. $K : \Omega \times E \to \Omega \times E$ is defined by

$$K(\omega, x) = \begin{cases} (\omega, \beta x) & x \in [0, a) \\ (\sigma(\omega), \beta x - \omega_1) & x \in [a, b] \\ (\omega, \beta x - 1) & x \in (b, (\beta-1)^{-1}]. \end{cases}$$

Given $(\omega, x) \in \Omega \times [a, b]$, the first return time is defined by

$$\tau(\omega, x) = \min\{n \geq 1 : K^i(\omega, x) \notin \Omega \times [a,b],\ 1 \leq i \leq n-1,\ K^n(\omega, x) \in \Omega \times [a,b]\}.$$

Define $K_{\Omega\times[a,b]}(\omega, x) = K^{\tau(\omega,x)}(\omega, x)$, and denote it for simplicity by I.

We now consider a special family of algebraic bases defined as follows. For any $n \geq 3$, let $1 < \beta < 2$ be the largest positive real number satisfying the equation

$$\beta^n = \beta^{n-2} + \beta^{n-3} + \cdots + \beta + 1.$$

The following lemma is clear.

Lemma 1.2. *For any $n \geq 3$, let $\beta_n > 1$ be the largest positive real root of the equation*

$$x^n = x^{n-2} + x^{n-3} + \cdots + x + 1. \tag{1}$$

Then (β_n) is an increasing sequence which converges to $2^{-1}(1+\sqrt{5})$.

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