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Topological conjugacy of constant length substitution dynamical systems

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Abstract

Primitive constant length substitutions generate minimal symbolic dynamical systems. In this article we present an algorithm which can produce the list of injective substitutions of the same length that generate topologically conjugate systems. We show that each conjugacy class contains infinitely many substitutions which are not injective. As examples, the Toeplitz conjugacy class contains three injective substitutions (two on two symbols and one on three symbols), and the length two Thue–Morse conjugacy class contains twelve substitutions, among which are two on six symbols. Together, they constitute a list of all primitive substitutions of length two with infinite minimal systems which are factors of the Thue–Morse system.

Keywords: Substitution dynamical system; Conjugacy; Sliding block code; Thue–Morse substitution; Toeplitz substitution

1. Prologue

In the article [6] published in 1971, the minimal dynamical systems arising from primitive substitutions on a binary alphabet having the same constant length were classified, yielding for

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a given such substitution a list of all substitutions of the same length generating topologically conjugate systems.

Topological conjugacy is the natural isomorphism notion for topological dynamical systems. Two systems (X, S) and (Y, T) are called *topologically conjugate* iff there exists homeomorphism $\psi : X \to Y$ such that $\psi \circ S = T \circ \psi$. The map ψ is called a *semi-conjugacy* if it is equivariant and merely continuous and onto. In this case (Y, T) is called a *factor* of (X, S).

In this paper we extend the classification of substitution dynamical systems to arbitrary finite alphabets. More recently, the articles [7,4] exhibit characterizations of such systems; these only implicitly yield corresponding topological conjugacies, and do not result in lists of conjugate systems. Also, in [17,19] a related goal has been partially accomplished – a classification of measure-theoretic conjugacy – for a restricted class of constant length substitutions.

If two constant length substitution systems are conjugate, then by Cobham's theorem, the lengths of the substitutions have a common non-zero integer power (see [5] for a short proof of the more simple version that we need here). Therefore, by taking suitable powers we can, and do, restrict our attention to substitutions of the same length L.

In this contribution we address the following two problems, in which L denotes a fixed integer larger than one, and σ is the left shift transformation.

Problem 1.1. Let α and β be two substitutions of the same length L, both primitive. Decide whether the dynamical systems (X_α, σ) and (X_β, σ) are topologically conjugate.

Problem 1.2. Let α be a primitive substitution of length L. Give a list of all the injective substitutions β of length L such that the dynamical systems (X_α, σ) and (X_β, σ) are topologically conjugate.

Finite systems are elementary, and we restrict attention everywhere to the non-periodic case of primitive substitutions with corresponding infinite minimal sets.

We show that to any primitive substitution of constant length whose minimal set is infinite, there are always infinitely many primitive substitutions of the same constant length having topologically conjugate minimal systems, but only finitely many of these are injective. Thus, the list produced by our algorithm for attacking Problem 1.2 will, starting from any given primitive substitution of constant length, consists of all injective substitutions of that length with dynamical systems topologically conjugate to the initial system. Clearly, since the list in Problem 1.2 is finite, Problem 1.1 has then also been solved, since there is a simple algorithm to associate to a substitution an injective substitution generating a conjugate system (cf. Section 6). This contrasts with the situation for the natural generalization of our problem to the collection of *all* substitutions. In [10] it is shown that there may be infinitely many primitive injective (non-constant length) substitutions that generate systems conjugate to a system generated by a substitution with the same Perron–Frobenius eigenvalue for its incidence matrix.

Recently a completely different solution has been obtained for Problem 1.1. in the paper [8]. Actually, because of Theorem 5.1, a solution of Problem 1.1 also yields a solution of Problem 1.2. However, it seems unfeasible – using the algorithm of [8] – to obtain the Thue–Morse list by hand, as we do in Section 11.

2. Substitutions and standard forms

We begin by recalling the basic definitions and known results without proof for primitive substitutions and their corresponding minimal systems, referring the reader to the standard Ref. [22].

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