INDAG: 431 Model 1 pp. 1–12 (col. fig: NIL)

ARTICLE IN PRESS

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Indagationes Mathematicae xx (xxxx) xxx–xxx

indagationes mathematicae

www.elsevier.com/locate/indag

Hankel matrices for the period-doubling sequence

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Abstract

We give an explicit evaluation, in terms of products of Jacobsthal numbers, of the Hankel determinants of order a power of two for the period-doubling sequence. We also explicitly give the eigenvalues and eigenvectors of the corresponding Hankel matrices. Similar considerations give the Hankel determinants for other orders.

1. Introduction

Let $\mathbf{s} = (s_n)_{n\geq 0}$ be a sequence of real numbers. The *Hankel matrix* $M_{\mathbf{s}}(k)$ of order k associated with $\mathbf{s}$ is defined as follows:

$$M_{\mathbf{s}}(k) = \begin{bmatrix} s_0 & s_1 & \cdots & s_{k-1} \\ s_1 & s_2 & \cdots & s_k \\ \vdots & \vdots & \ddots & \vdots \\ s_{k-1} & s_k & \cdots & s_{2k-2} \end{bmatrix}. \tag{1}$$

See, for example, [17]. Note that the rows of $M_{\mathbf{s}}(k)$ are made up of successive length-k "windows" into the sequence $\mathbf{s}$.

Of particular interest are the determinants $\Delta_{\mathbf{s}}(k) = \det M_{\mathbf{s}}(k)$ of the Hankel matrices in (1), which are often quite challenging to compute explicitly. In some cases when these determinants are non-zero, they permit estimation of the irrationality measure of the associated real numbers $\sum_{n\geq 0} s_n b^{-n}$, where $b \geq 2$ is an integer; see, for example, [2,7,5,6,26,19,4]. In some sense, the

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http://dx.doi.org/10.1016/j.indag.2016.11.008

Please cite this article in press as: R.J. Fokkink, et al., Hankel matrices for the period-doubling sequence, Indagationes Mathematicae (2016), http://dx.doi.org/10.1016/j.indag.2016.11.008

Hankel determinants measure how "far away" the sequence $\mathbf{s}$ is from a linear recurrence with constant coefficients, since for such a sequence we have $H_{\mathbf{s}}(n) = 0$ for all sufficiently large n.

In this note we consider the Hankel determinants for a certain infinite sequence of interest, the so-called *period-doubling sequence*

$$\mathbf{d} = (d_i)_{i\geq 0} = 1011101010111011101110101011101 \cdots .$$

This sequence can be defined in various ways [8], but probably the three simplest are as follows:

- as the fixed point of the map
$$1 \to 10, \qquad 0 \to 11;$$
- as the first difference, taken modulo 2, of the Thue–Morse sequence $\mathbf{t} = 0110100110010110\cdots$ (fixed point of the map $0 \to 01$, $1 \to 10$);
- as the sequence defined by
$$d_i = \begin{cases} 1, & \text{if } s_2(i) \not\equiv s_2(i+1) \pmod 2; \\ 0, & \text{otherwise;} \end{cases}$$
where $s_2(i)$ is the sum of the binary digits of i when expressed in base 2.

We explicitly compute the Hankel determinants when the orders are a power of 2, and we also compute the eigenvalues and eigenvectors of the corresponding Hankel matrices. We derive recursions for Hankel determinants for all orders. Finally, we also consider the determinants for the complementary sequence

$$\overline{\mathbf{d}} = 0100010101000100010001010100010 \cdots ,$$

obtained from $\mathbf{d}$ by changing 1 to 0 and vice versa.

1.1. Previous work

By considering $\Delta_{\mathbf{d}}(n)$ modulo 2, Allouche, Peyrière, Wen, and Wen [1, Prop. 2.2] proved that $\Delta_{\mathbf{d}}(n)$ is odd for all $n \geq 1$. However, they did not obtain any explicit formula for $\Delta_{\mathbf{d}}(n)$. In fact, their main focus was on the non-vanishing of the Hankel determinants for the Thue–Morse sequence on values ± 1. For this, also see Bugeaud and Han [3] and Han [14]. Recently Fu and Han [11] also studied some Hankel matrices associated with the period-doubling sequence, but they did not obtain our result.

There are only a small number of sequences defined by iterated morphisms for which the Hankel determinants are explicitly known (even for subsequences). These include the infinite Fibonacci word [18], the paperfolding sequence [13,11], the iterated differences of the Thue–Morse sequence [12], the Cantor sequence [27], and sequences related to the Thue–Morse sequence [15,9,10].

2. Hankel determinants

Here are the first few terms of the Hankel determinants for the period-doubling sequence and its complementary sequence:

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\Delta_{\mathbf{d}}(k)$	1	1	−1	−3	1	1	−1	−15	1	1	−1	−3	1	1	−9	−495
$\Delta_{\overline{\mathbf{d}}}(k)$	0	−1	0	1	0	−1	0	9	0	−1	0	1	0	−1	0	225