ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Indagationes Mathematicae ▮ (▮▮▮▮) ▮▮▮–▮▮▮

indagationes mathematicae

www.elsevier.com/locate/indag

# On the Frobenius problem for Beatty sequences

Jörn Steuding*, Pascal Stumpf

*Department of Mathematics, Würzburg University, Emil-Fischer-Str. 40, 97 074 Würzburg, Germany*

**Abstract**

We study the solvability of linear diophantine equations in two variables within the context of Beatty sequences.


*Keywords:* Linear diophantine equation; Frobenius problem; Beatty sequence

## 1. The main results

Let $a, b, c$ be positive integers. A classical result due to Bézout claims that the linear diophantine equation

$$aX \pm bY = c$$

is solvable (for both cases of the sign '±') within the set of integers if and only if the greatest common divisor of $a$ and $b$ divides $c$. The set of solutions can easily be determined by applying the euclidean algorithm to $a$ and $b$ (see Hardy & Wright [6]); the sign is not relevant since there is an obvious bijection between the respective sets of solutions. However, the question about solutions in *nonnegative* integers with a plus sign '+' is slightly more difficult. In order to exclude the trivial case of non-solvability we may assume that $a$ and $b$ are coprime. It is well-known that for every positive integer

$$c > ab - a - b \tag{1}$$

there exist $x, y \in \mathbb{N}_0$ such that $ax+by = c$, whereas for $c = ab-a-b$ there is no solution within $\mathbb{N}$. This as well as generalizations to linear diophantine equations in more than two variables has

* Corresponding author.
*E-mail addresses:* steuding@mathematik.uni-wuerzburg.de (J. Steuding), littlefriend@mathlino.org (P. Stumpf).

http://dx.doi.org/10.1016/j.indag.2016.11.020

been investigated by several mathematicians, the most prominent early contributors from the 19th century being Frobenius and Sylvester. Accordingly, this topic has been investigated under the name Frobenius problem (or coin problem or stamp problem). Surprisingly, already for the case of four variables there is no closed formula such as (1) for the largest $c$ that is not representable. For this and further information on the original problem and the recent state of the art we refer to Ramirez Alfonsin [8].

In this short note we only treat the case of two variables and the additional restriction that $x$ and $y$ shall be elements of Beatty sequences.

Given a positive real number $\alpha$ the associated Beatty sequence (resp. Beatty set) is defined by

$$\mathcal{B}(\alpha) = \{\lfloor n\alpha \rfloor \,:\, n \in \mathbb{N}\},$$

where we write $x = \lfloor x \rfloor + \{x\}$ with $\lfloor x \rfloor$ denoting the largest integer less than or equal to $x$ and $\{x\}$ is the fractional part. For example, we have $\mathcal{B}(\alpha) = \mathbb{N}$ (as a set) for every $\alpha \leq 1$; however, if $\alpha > 1$, then $\mathcal{B}(\alpha)$ is a proper subset of $\mathbb{N}$ of positive density $\frac{1}{\alpha}$. The name is misleading. Beatty sequences already appeared in the works of Christoffel [4] and Strutt (a.k.a. Lord Rayleigh) [10]; it was Beatty who popularized the topic by a problem he posed in 1926 in the American Mathematical Monthly [1,2]. The proposed task was to show that $\mathcal{B}(\alpha) \cup \mathcal{B}(\beta) = \mathbb{N}$ is a disjoint union for irrational $\alpha$ and $\beta$ related to one another by $\frac{1}{\alpha} + \frac{1}{\beta} = 1$; this is now well-known as both, Beatty's theorem and Rayleigh's theorem.

Although Beatty sequences may exclude quite many integers, they exhibit sufficiently many structures to find integer solutions to linear diophantine equations under a natural diophantine condition:

**Theorem 1.** *Let $a, b \in \mathbb{N}$ be coprime and $\alpha, \beta \in \mathbb{R}_{>1}$. Then, for every $c \in \mathbb{N}$, the equation*

$$aX - bY = c$$

*is solvable with a solution $x \in \mathcal{B}(\alpha)$ and $y \in \mathcal{B}(\beta)$ if $1, \frac{1}{\alpha}, \frac{1}{\beta}$ are linearly independent over $\mathbb{Q}$. In this case there exist infinitely many solutions and the set of solving $x$ (resp. $y$) forms a set of positive density within $\mathbb{N}$.*

**Theorem 2.** *Let $a, b \in \mathbb{N}$ be coprime and $\alpha, \beta \in \mathbb{R}_{>1}$. Then, for every sufficiently large $c \in \mathbb{N}$, the equation*

$$aX + bY = c$$

*is solvable with a solution $x \in \mathcal{B}(\alpha)$ and $y \in \mathcal{B}(\beta)$ if $1, \frac{1}{\alpha}$ and $\frac{1}{\beta}$ are linearly independent over $\mathbb{Q}$.*

Notice that the equations $X \pm Y = 1$ are not solvable within $2\mathbb{N} = \mathcal{B}(2)$. This already shows that a diophantine condition on $\alpha$ and $\beta$ is necessary. In fact, the linear independence of $1, \frac{1}{\alpha}$ and $\frac{1}{\beta}$ over $\mathbb{Q}$ allows us to use a multidimensional version of Kronecker's inhomogeneous approximation theorem [7] and Weyl's refinement [3] in our proofs. Therefore, our reasoning differs from the classical case. It is an interesting problem whether the diophantine conditions are necessary. In the special case $a = b = 1$ we are able to remove the linear independence condition from Theorem 2 as follows from:

**Theorem 3.** *For any $1 \leq \alpha, \beta < 2$ the following set is finite:*

$$\mathbb{N} \setminus (\mathcal{B}(\alpha) + \mathcal{B}(\beta)).$$