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Growth and homogeneity of matchbox manifolds

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Abstract

A matchbox manifold with one-dimensional leaves which has equicontinuous holonomy dynamics must be a homogeneous space, and so must be homeomorphic to a classical Vietoris solenoid. In this work, we consider the problem, what can be said about a matchbox manifold with equicontinuous holonomy dynamics, and all of whose leaves have at most polynomial growth type? We show that such a space must have a finite covering for which the global holonomy group of its foliation is nilpotent. As a consequence, we show that if the growth type of the leaves is polynomial of degree at most 3, then there exists a finite covering which is homogeneous. If the growth type of the leaves is polynomial of degree at least 4, then there are additional obstructions to homogeneity, which arise from the structure of nilpotent groups.

Keywords: Non-abelian group actions on Cantor sets; Growth of leaves; Group chains; Regularity; Homogeneity

1. Introduction

A *continuum* is a compact connected metric space. A continuum X is *homogeneous* if for each $x, y \in X$, there exists a homeomorphism $h: X \to X$ such that $h(x) = y$. For example, a compact connected manifold without boundary is a homogeneous continuum, and the proof of this is a standard exercise in manifold theory. On the other hand, Knaster and Kuratowski [26] posed the problem in 1920 to classify the homogeneous continua in the plane $\mathbb{R}^2$, and this problem

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was only recently solved by Hoehn and Oversteegen [23]. Their paper also gives a selection of references for the "rich literature concerning homogeneous continua". For a continuum embedded in Euclidean space $\mathbb{R}^n$ with $n > 2$, the classification problem for homogeneous continua becomes intractable in this full generality, so one formulates a more restricted problem by imposing conditions on the continua.

In this work, we are concerned with continua that are "manifold-like". That is, the continuum is a disjoint union of manifolds of the same dimension, and locally has a uniform structure. This is formulated by introducing the notion of an *n-dimensional foliated space* $\mathfrak{M}$, which is a continuum that has a regular local product structure [6,32]; that is, every point $x \in \mathfrak{M}$ has an open neighborhood $x \in U \subset \mathfrak{M}$ homeomorphic to an open subset of $\mathbb{R}^n$ times a compact metric space $\mathfrak{T}_x$ where $\mathfrak{T}_x$ is called the local transverse model. The homeomorphism $\varphi_x : \overline{U}_x \to [-1,1]^n \times \mathfrak{T}_x$ is called a local foliation chart. A *matchbox manifold* is a foliated space $\mathfrak{M}$ such that the local transverse models $\mathfrak{T}_x$ are totally disconnected. The leaves of the foliation $\mathcal{F}$ of $\mathfrak{M}$ are the maximal path connected components, and by assumption they are smooth manifolds, and can be endowed with complete Riemannian metrics. Precise definitions can be found in the Refs. [2,6,8,9].

Bing conjectured in [5] that a homogeneous continuum whose arc-composants are arcs must be a classical Van Dantzig–Vietoris solenoid [39,40]. This condition is satisfied by 1-dimensional matchbox manifolds, so in particular Bing's conjecture implies that such spaces are solenoids if homogeneous. An affirmative answer to this conjecture of Bing was given by Hagopian [21], and subsequent proofs in the framework of 1-dimensional matchbox manifolds were given by Mislove and Rogers [30] and by Aarts, Hagopian and Oversteegen [1].

Clark and Hurder generalized the 1-dimensional result to higher dimensional leaves in the work [8], giving a positive solution to Conjecture 4 of [17]. In Section 2, we recall the notion of weak solenoids, and the special case of normal (or McCord) solenoids, as introduced by McCord in [28]. Then the following result was proved in [8]:

Theorem 1.1 (*Clark & Hurder*)**.** *Let $\mathfrak{M}$ be a homogeneous matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to a McCord solenoid.*

One step in the proof of Theorem 1.1 is the proof of the following result, which generalizes a key step in the proofs of the 1-dimensional case in [1,30]. A foliation $\mathcal{F}$ is said to be *equicontinuous* if the transverse holonomy pseudogroup defined by the parallel transport along the leaves of $\mathcal{F}$ acting on a transversal space has equicontinuous dynamics. (See [8] for a detailed discussion of this property.)

Theorem 1.2 (*Clark & Hurder [8]*)**.** *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to a weak solenoid.*

The assumption that the holonomy action is equicontinuous is essential. The Williams solenoids, as defined in [41] as inverse limits of maps between branched n-manifolds, are matchbox manifolds whose holonomy dynamics is expansive, and they are not homeomorphic to a weak solenoid.

Examples of Schori [37] and Rogers and Tollefson [35] show that there exists weak solenoids that are not homogeneous. The paper of Fokkink and Oversteegen [17] analyzed the problem of showing that a weak solenoid is in fact homogeneous, introducing a technique based on the group chains associated to a weak solenoid, to obtain a criterion for when a weak solenoid is homogeneous.

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