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The principal moments of inertia calculated with the hydrostatic equilibrium figure of the Earth

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ABSTRACT

As an indication of the Earth's mass distribution, the principal moments of inertia (PMOI, i.e., *A, B, C*) of the Earth are the basic parameters in studies of the global dynamics of the earth, like earth nutation, and the geophysics. From the aspect of observation, the PMOI can be calculated from the spherical coefficients of observed gravity field. In this paper, the PMOI are calculated directly according to its definition with the figures of the Earth's interior derived by a generalized theory of the hydrostatic equilibrium figure of the Earth. We obtain that the angle between the principal axis of the maximum moment of PMOI and the rotational axis is 0.184°, which means that the other two principal axes are very closely in the equatorial plane. Meanwhile, *B-A* is $1.60 \times 10^{-5}\ MR^2$, and the global dynamical flattening (H) is calculated to be 3.29587×10^{-3}, which is 0.67% different from the latest observation derived value $H_{obs}(3.273795 \times 10^{-3})$ (Petit and Luzum, 2010), and this is a significant improvement from the 1.1% difference between the value of H derived from traditional theories of the figure of the Earth and the value of $H_{obs.}$ It shows that we can calculate the PMOI and H with an appropriate accuracy by a generalized theory of the hydrostatic equilibrium figure of the Earth.

1. Introduction

The principal moments of inertia (PMOI) of the Earth (*A, B, C*) are the measurement of the Earth's rotational inertia and the geodetic and astronomical fundamental parameters. They are most fundamentally rooted in Earth's rotational dynamics, as an indication of the overall distribution of mass and density in the Earth's interior.

The PMOI can be calculated by using global gravity field coefficients which can be obtained nowadays through satellite observations like GRACE mission etc., and it is clearly true that assumptions of hydrostatic equilibrium (HSE) are not always required. Liu & Chao [2] obtain the results that the axis of the least moment of inertia (A) lies along the (14.93°W, 165.07°E) diameter and the difference *B-A* is $7.26 \times 10^{-6}\ MR^2$. Chen & Shen [3] and Chen [4] obtain that the axis of the least moment of inertia (A) lies along the (14.929°W, 165.071°E) diameter. For rapidly rotating celestial bodies, like the Earth, the rate of precession of the spin axis provides information which complements the J_2 and $C_{2,2}$ estimates, and allows a determination of the PMOI without assuming HSE. Dehant and Capitaine [5] gives a summary of obtaining the value of H from various kinds of observations including earth precession. The International Astronomy Union (IAU) and International Earth Rotation and reference from Services (IERS) give the recommended value of H as 3.273795×10^{-3} or approximately 1/305.45 [1].

In fact, the figure of the Earth's interior is one of the fundamental issues in solid earth sciences today. The traditional theories of studying the figure of the Earth's interior usually start from HSE and one dimensional Earth model, like PREM [6], obtaining the equi-potential surface with symmetrical shape. The first generation of such kind theories is the famous Clairaut equation [7], in which J_2 term is considered only. A small correction to Clairaut equation, i.e.,

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a J_4 term, is added in the Darwin–de Sitter theory [8,9]. And the third generation of these theories is proposed and completed by Prof. C. Dennis [10]. The most significant common characteristics of these theories is that the Earth is assumed rigorously as rotational symmetry and equatorial symmetry, the result of which is that $A = B$.

In this paper, the equi-potential figures of the Earth's interior are calculated with a new generalized theory, proposed by Liu & Huang [11] and Liu [12], of the figure of the Earth to third-order terms of order f^3 (f being the geometric oblateness as usual) in Section 2. In this theory, both the direct and indirect contributions of the anti-symmetric crust layer are included while considering the topographic existence of the crust and its gravitational effects on the equi-potential figure of the Earth's interior.

We separate the Earth into two parts from the crust-mantle boundary to solve the moments of inertia and products of inertia of the Earth in Section 3. And then the numerical values of the PMOI are presented by the method of solving the homogeneous linear algebraic equations in Section 4, based on the data of the figure of the Earth's interior providing in Section 2.

In Section 5, we analyze the conclusions and show that we can calculate the PMOI and H with an appropriate accuracy by a generalized theory of the hydrostatic equilibrium figure of the Earth.

2. A generalized theory of the figure of the Earth to third-order terms of order f^3

In the model of isotropic Earth with constant rotation and in hydrostatic equilibrium (HSE), the surfaces of the equi-density, the equi-potential and the equi-pressure interior coincide with each other. The traditional formula to express this equi-potential surface is

$$r(s,\theta) = s\left[1 + \sum_{n=0}^{\infty} s_{2n}(s)P_{2n}(\cos\theta)\right] \tag{1}$$

where r is the distance between a point $P(r,\theta,\lambda)$ in the Earth and the geocenter, s is the mean equi-volumetric radii of this equi-potential surface crossing the point P, θ and λ are the colatitude and longitude, $P_{2n}(cos\theta)$ is the Legendre function. The problem of finding the figure of the Earth thus becomes the problem of determining an infinite set of functions $s_{2n}(s)$. The summation over n is usually, and in practice, truncated to 1, 2, or 3, which are the cases of the theories of Clairaut, Dawin-de Sitter, and Dennis respectively, and their precision are of the first, second and third order of ellipticity, respectively.

For the anisotropic Earth, the above equation can be written in a general form as follow:

$$r(s,\theta,\lambda) = s\left[1 + \sum_{n=0}^{\infty}\sum_{m=-n}^{n} H_n^m(s)Y_n^m(\theta,\lambda)\right] \tag{2}$$

where

$$Y_n^m(\theta,\lambda) = (-1)^m\left(\frac{2n+1}{4\pi}\right)^{1/2}\left[\frac{(n-m)!}{(n+m)!}\right]^{1/2} P_n^m(\cos\theta)e^{im\lambda} \tag{3}$$

λ is the longitude, $P_n^m(\cos\theta)$ is associated Legendre function of degree n and order m, $H_n^m(s)$ should be solved in the follow.

Hence, the gravity potential, defined as the sum of the Newtonian gravitational potential (V) and centrifugal potential (Z), can then be derived into following form [11].

$$\begin{aligned} W(r) &= V(r) + Z(r) \\ &= \frac{GM}{r} + \frac{1}{2}r^2\omega^2\sin^2\theta \\ &= GE_0(s) + G\bar{\rho}s^2\sum_{n=1}^{\infty}\sum_{m=-n}^{n} Y_n^m(\theta,\lambda)\Xi_n^m(\theta,\lambda) \end{aligned} \tag{4}$$

which gives

$$\Xi_n^m = m_h p_{n,m} + \sum_{l=0}^{\infty}\frac{s^{l-2}}{\bar{\rho}}u_{l,n,m} + \sum_{l=1}^{\infty} g_{l,n,m} + \sum_{l=0}^{\infty} f_{l,n,m} \tag{5}$$

where G is the gravitational constant, M is the total Earth's mass, ω is the angular speed of the rotation of the Earth, p, u, g and f are the implicit functions of $H_n^m(s)$.

By definition, the gravity potential W Over the equi-potential surfaces must be a function of s only and independent on (θ,λ).

Thus we have

$$\begin{cases} \Xi_n^m + (-1)^m\Xi_n^{-m*} = 0 \\ n = 1,\ldots,\infty \\ m = 0,\ldots,n \end{cases} \tag{6}$$

Here n is truncated at 6, that is to say, it is at the third-order approximation. We can then obtain all the parameters $H_n^m(s)$ of all the equi-potential surfaces of s (starting from near geocenter to the Earth's surface) of the Earth's interior with formula (2).

3. The moment of inertia of the Earth

In principle the MOI can be calculated respect to any axes. Here, let X, Y and Z be the coordinate system with its origin at the geocenter, in which axis Z is the Earth's rotational axis and axis X points to the prime meridian. Let the PMOI of the Earth be A, B and C, where $A < B < C$, and the corresponding principal axes of inertia be axes a, b and c. Since the Earth is not a regular sphere completely, the Earth's principal axis (c) is inconsistent with the rotational axis (Z) (see Fig. 1) [2].

The MOI of a rigid body rotation around any axis that passes its origin can be written as follow:

$$\begin{aligned} I = {} & I_{xx}\cos^2\alpha + I_{yy}\cos^2\beta + I_{zz}\cos^2\gamma - 2I_{xy}\cos\alpha\cos\beta \\ & - 2I_{yz}\cos\beta\cos\gamma - 2I_{zx}\cos\gamma\cos\alpha \end{aligned} \tag{7}$$

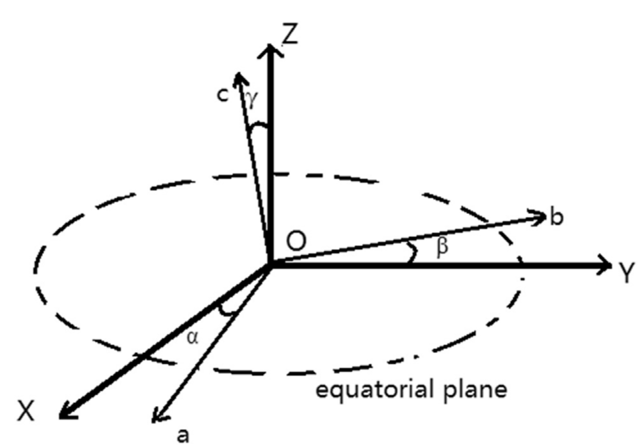

Fig. 1. The relationship between the Earth's principal axes (a, b, c) and rotational axes (X, Y, Z).

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