Physics and Chemistry of the Earth 98 (2017) 3–15

Contents lists available at ScienceDirect

Physics and Chemistry of the Earth

journal homepage: www.elsevier.com/locate/pce

From standard to fractional structural visco-elastodynamics: Application to seismic site response

C. Germoso [a], A. Fraile [a], E. Alarcon [a], J.V. Aguado [b], F. Chinesta [b,*]

[a] School of Industrial Engineering (ETSII), Universidad Politécnica de Madrid (UPM), José Gutiérrez Abascal, 28006 Madrid, Spain
[b] Institut de Recherche en Genie Civil et Mécanique (GeM UMR CNRS 6183), Ecole Centrale de Nantes. 1 rue de la Noe, BP 92101, F-44321 Nantes cedex 3, France

ARTICLE INFO

Article history:
Received 16 October 2015
Received in revised form
22 December 2015
Accepted 30 January 2016
Available online 12 February 2016

Keywords:
Elastodynamics
Generalized Kelvin-Voigt model
Fractional derivatives
Seismic site analysis
Soil mechanics

ABSTRACT

This paper revisits visco-elastodynamics from its most standard formulation to some more advanced description involving frequency-dependent damping (or viscosity), analyzing the effects of considering fractional derivatives for representing such viscous contributions. We will prove that such a choice results in richer models that can accommodate different constraints related to the dissipated power, response amplitude and phase angle. Moreover, the use of fractional derivatives allows to accommodate in parallel, within a generalized Kelvin-Voigt analog, many dashpots that contribute to increase the modeling flexibility for describing experimental findings. Finally, the effect of fractional damping in dynamic soil models will be addressed within a seismic site analyses framework.

1. Introduction to standard computational dynamics

Solid dynamics is usually formulated either in the time or in the frequency domains. The first is preferred when calculating transient responses, whereas the frequency approach is an appealing alternative for calculating forced responses, both extensively used and described in many reference books, as for example (Clough and Penzien, 1993). The general discrete form of linear solid dynamics writes

$$\mathbf{M}\frac{d^2\mathbf{U}(t)}{dt^2}+\mathbf{C}\frac{d\mathbf{U}(t)}{dt}+\mathbf{K}\mathbf{U}(t)=\mathbf{F}(t), \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are respectively the mass, damping and stiffness matrices, $\mathbf{U}$ the vector that contains the nodal displacements and $\mathbf{F}$ the nodal excitations (forces).

The main drawback related to the time integration of Eq. (1) lies in the necessity of solving a linear system (usually of very large size) at each time step, in particular when some of these matrices change in time for a variety of reasons (time dependent behavior, non-linearities, …).

Loads can be easily expressed in the frequency domain. In what follows we consider without loss of generality the simplest scenario: $\mathbf{F}(t) = \mathbf{f}g(t)$, with $||\mathbf{f}|| = 1$. The time function $g(t)$ can be expressed from the superposition of harmonic functions $e^{i\omega t}$, with ω the circular frequency and $i = \sqrt{-1}$. If we assume a single frequency harmonic excitation, $g(t) = e^{i\omega t}$, the response of a linear solid is expected having the same frequency but exhibiting a certain phase angle θ, i.e. $\mathbf{U}(t) = \overline{\mathbf{U}}e^{i\omega t+i\theta}$, where $\overline{\mathbf{U}}$ is the vector containing the amplitude of the nodal displacements. This vector can be rewritten as $\mathbf{U}(t) = \overline{\mathbf{U}}e^{i\omega t+i\theta} = \mathbb{U}e^{i\omega t}$, where now $\mathbb{U} = \overline{\mathbf{U}}e^{i\theta}$ denotes a vector of complex entries, with $\mathbb{U} = \mathbf{U}_r + i\mathbf{U}_i$, where $\mathbf{U}_r$ and $\mathbf{U}_i$ are respectively the real and imaginary parts of $\mathbb{U}$.

By introducing $\mathbf{F}(t) = \mathbf{f}e^{i\omega t}$ and $\mathbf{U}(t) = \mathbb{U}e^{i\omega t}$ into Eq. (1) it results the frequency-based description of solid dynamics

$$\left(-\omega^2\mathbf{M}+i\omega\mathbf{C}+\mathbf{K}\right)\mathbb{U}=\mathbf{f}, \tag{2}$$

where the exponential factor $e^{i\omega t}$ was eliminated from both members.

If damping vanishes, i.e. $\mathbf{C} = 0$, and one focuses on the free response of the mechanical system, i.e. $\mathbf{f} = 0$, then Eq. (2) reduces to:

* Corresponding author.
E-mail address: Francisco.Chinesta@ec-nantes.fr (F. Chinesta).

http://dx.doi.org/10.1016/j.pce.2016.01.005
1474-7065/

$$\mathbf{K}\mathbb{U} = \omega^2\mathbf{M}\mathbb{U}, \tag{3}$$

that defines an eigenproblem that results in the eigenmodes $\mathbb{U}_i$ and the associated eigenfrequencies ω_i^2. Eigenmode $\mathbb{U}_i$ scaled from some normalization condition is called normal mode and is noted by ϕ_i. It is usual to normalize eigenmodes according to $\phi_i^T\mathbf{M}\phi_i = M_i = 1$, from which it results $\phi_i^T\mathbf{K}\phi_i = K_i = \omega_i^2$, where M_i and K_i are known as modal mass and modal stiffness respectively. If normal modes are placed in the columns of matrix $\mathbf{P}$, we could express $\mathbb{U}$ in the orthonormal basis defined by the normal modes, according to

$$\mathbb{U} = \mathbf{P}\cdot\xi(t). \tag{4}$$

Now, by injecting (4) into Eq. (1), premultiplying by the transpose of $\mathbf{P}$ and taking into account the orthogonality conditions $\phi_j^T\mathbf{M}\phi_i = 0$ and $\phi_j^T\mathbf{K}\phi_i = 0$ when $i \neq j$, it results

$$\mathbf{I}\frac{d^2\xi(t)}{dt^2} + \mathbf{P}^T\mathbf{C}\mathbf{P}\frac{d\xi(t)}{dt} + \mathbf{diag}\left(\omega_i^2\right)\xi(t) = \mathbf{P}^T\mathbf{F}(t) \tag{5}$$

where $\mathbf{I}$ is the unit matrix.

When damping vanishes, $\mathbf{C} = 0$, the previous equation reduces to a linear system of uncoupled second order ordinary differential equations.

When damping applies matrix $\tilde{\mathbf{C}} \equiv \mathbf{P}^T\mathbf{C}\mathbf{P}$ is not in general diagonal compromising the efficiency of modal analysis. To circumvent this issue different diagonalization procedures have been proposed and widely used. Two usual diagonalization procedures are: (i) diagonalization by model damping that expresses $\tilde{\mathbf{C}} = \mathbf{diag}(2\zeta_i\omega_i)$, where ζ_i denotes the damping ratio for the i-th natural mode; and (ii) Rayleigh diagonalization that by assuming $\mathbf{C} = a_0\mathbf{M} + a_1\mathbf{K}$ results in $\tilde{\mathbf{C}} = \mathbf{diag}(a_0 + a_1\omega_i^2) = \mathbf{diag}(2\zeta_i\omega_i)$, with $\zeta_i = 1/2(a_0/\omega_i + a_1\omega_i)$. These choices imply approximations whose validity and accuracy must be checked.

A more precise route consists of extracting the modes from the solution of the quadratic complex eigenproblem

$$\left(\mathbf{K} + \mathrm{i}\omega\mathbf{C} - \omega^2\mathbf{M}\right)\mathbb{U} = 0. \tag{6}$$

However, many times models involves parametric damping, that is, damping depends on some parameters grouped in vector $\boldsymbol{\mu}$, $\mathbf{C}(\boldsymbol{\mu})$, and in that case the solution of parametric quadratic eigenproblems remains an open issue (Quraishi et al., 2014) (Tisseur and Meerbergen, 2001).

When one is interested in solving problems with parametric damping, the best choice, in our opinion, is renouncing to direct time integrations and also to modal analysis based time integrations, in favor of an alternative approach, purely harmonic, making use of Eq. (2).

In what follows we assume that the applied load can be written from the superposition of harmonic functions of angular frequency ω

$$g(t) = \int_{-\infty}^{\infty} \mathcal{G}(\omega)e^{\mathrm{i}\omega t}d\omega, \tag{7}$$

where $\mathcal{G}(\omega)$ represents the content of each harmonic $e^{\mathrm{i}\omega t}$ in $g(t)$. In fact $\mathcal{G}(\omega)$ is the Fourier transform of $g(t)$

$$\mathcal{G}(\omega)\equiv\mathcal{F}(g(t)) = \int_{-\infty}^{\infty} g(t)e^{-\mathrm{i}\omega t}dt. \tag{8}$$

In general $\mathcal{G}(\omega<\omega^-) = \mathcal{G}(\omega>\omega^+)\approx 0$, that is

$$g(t)\approx \int_{\omega^-}^{\omega^+} \mathcal{G}(\omega)e^{\mathrm{i}\omega t}d\omega, \tag{9}$$

that implies that Eq. (2) must be solved for any value of $\omega\in[\omega^-,\omega^+]$

$$\left(-\omega^2\mathbf{M} + \mathrm{i}\omega\mathbf{C} + \mathbf{K}\right)\mathbb{U}(\omega) = \mathbf{f}, \tag{10}$$

that leads to the parametric solution $\mathbb{U}(\omega)$, that by applying the superposition principle that characterizes linear behaviors leads to the general solution

$$\mathbf{U}(t) = \int_{\omega^-}^{\omega^+} \mathcal{G}(\omega)\mathbb{U}(\omega)e^{\mathrm{i}\omega t}d\omega. \tag{11}$$

The main drawback of that approach is the necessity of solving a linear system related to the solution of Eq. (10) for each value of ω involved in the discrete inverse transform (11), number that increases with the frequency interval length $\Delta\omega = |\omega^+ - \omega^-|$ and with the signal resolution. For this reason, modal analysis is much more employed than harmonic analysis.

In the general parametric case, mass, damping and stiffness matrices can depend on a series of parameters grouped in the vector $\boldsymbol{\mu}$, i.e. $\mathbf{M}(\boldsymbol{\mu})$, $\mathbf{C}(\boldsymbol{\mu})$ and $\mathbf{K}(\boldsymbol{\mu})$, making difficult, as indicated above, the employ of modal analysis that requires the solution of parametric eigenproblems (Quraishi et al., 2014) (Tisseur and Meerbergen, 2001). On the other hand the use of harmonic analysis requires solving Eq. (10) for each frequency ω and each possible choice of the parameters $\boldsymbol{\mu}_j$, $\mathbb{U}(\omega;\boldsymbol{\mu}_j)$ to finally compute the discrete sum related to

$$\mathbf{U}(t;\boldsymbol{\mu}_j) = \int_{\omega^-}^{\omega^+} \mathcal{G}(\omega)\mathbb{U}(\omega;\boldsymbol{\mu}_j)e^{\mathrm{i}\omega t}d\omega \tag{12}$$

for any choice of the parameters $\boldsymbol{\mu}_j$.

Thus, if for example we consider two parameters $\boldsymbol{\mu}^T = (\mu_1, \mu_2)$, each one sampled using hundred values, $\boldsymbol{\mu}_j$ involves 10^4 samples, i.e. $j = 1,\cdots,10^4$. Now, if we assume 10^4 discrete frequencies involved in the reconstruction of $g(t)$, the calculation of the parametric solution $\mathcal{U}(\omega_i;\boldsymbol{\mu}_j)$ requires solving 10^8 linear systems.

The use of the Proper Generalized Decomposition largely considered in our former works (Chinesta et al., 2011, 2010, 2013, 2014), allows solving the parametric model

$$\left(-\omega^2\mathbf{M}(\boldsymbol{\mu}) + \mathrm{i}\omega\mathbf{C}(\boldsymbol{\mu}) + \mathbf{K}(\boldsymbol{\mu})\right)\mathbb{U}(\omega,\boldsymbol{\mu}) = \mathbf{f} \tag{13}$$

by assuming the separated representation

$$\mathbb{U}(\omega,\mu_1,\mu_2)\approx\sum_{k=1}^{N}\mathbf{Z}_k W_k(\omega)\mathcal{M}_k^1(\mu_1)\mathcal{M}_k^2(\mu_2), \tag{14}$$

where $\mathbf{Z}_k$ is a vector of nodal displacements and $W_k(\omega)$, $\mathcal{M}_k^1(\mu_1)$ and $\mathcal{M}_k^2(\mu_2)$ are functions that depend on the extra-coordinates ω, μ_1 and μ_2, respectively. The construction of the separated representation (14) implies the solution of a number of linear systems scaling with the number of terms involved in the finite sum, i.e. in the order of N linear systems (N being in general of few tens).

In soil mechanics (Pecker, 1984) the damping is assumed scaling with the inverse of frequency. The interested reader can refer to (Crandall, 1970) that analyzed the theoretical consequences of assuming a frequency dependent dashpot parameter. In (Crandall,