Sci. Bull. (2016) 61(7):570–575
DOI 10.1007/s11434-016-1040-z

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Article

**Physics & Astronomy**

# Localization and shock waves in curved manifolds

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Received: 15 September 2015 / Revised: 22 February 2016 / Accepted: 24 February 2016 / Published online: 10 March 2016


**Abstract** The investigation of the interplay between geometry and nonlinearity may open the road to the control of extreme waves. We study three-dimensional localization and dispersive shocks in a bent cigar shaped potential by the nonlinear Schrödinger equation. At high bending and high nonlinearity, topological trapping is frustrated by the generation of curved wave-breaking. Four-dimensional parallel simulations confirm the theoretical model. This work may contribute to novel devices based on geometrically constrained highly nonlinear dynamics and tests and analogs of fundamental physical theories in curved space.

**Keywords** Nonlinear waves · Shock waves · Nonlinear optics · Curvature · Bose–Einstein condensation

## 1 Introduction

Lord Rayleigh and others studied the effect of geometry on wave propagation since early developments of the theory of sound, for example, in the vibrations of membranes with different shapes [1]. Geometry changes energy velocity, wave transformations upon propagation, and various classical and quantum undulatory phenomena. This effect occurs in constrained micron-scale propagation and at an astrophysical scale. Renowned examples include the Einstein lensing effect [2], the Hawking radiation [3], or the Unruh effect [4].

Notwithstanding the old history, the link between geometry and waves is continuously fascinating many researchers, and is currently driving important applications as transformation optics [5, 6], curved and twisted waveguides [5, 7] and analogs of gravity in Bose–Einstein condensation (BEC) [8, 9] or nonlinear optics [10–13].

Geometry not only changes linear propagation; studies on solitons in curved manifolds reveal many interesting phenomena on nonlinear waves [14–19]. Topologically induced localization is perhaps the most striking [20]: constraining the wave in extremely deformed regions, inhibits propagation and traps energy at the largest curvature. We do not know, however, if geometrical localization can compete with nonlinearity, and the dynamics resulting from a nonlinear response that overcomes topological bounds. We may expect that at high nonlinearity shock waves (SW) arise [21]. Indeed, recent studies in optics and Bose–Einstein condensation (BEC) [22–27] reveal SW from singular solutions of the hydrodynamic reduction of the nonlinear Schroedinger equation (NLS) that are regularized by oscillating wave fronts ("dispersive SW", D-SW); but we do not know the effect of curved space on DSW. Understanding the way geometry alters highly nonlinear regimes may allow controlling and engineering wave-breaking and related phenomena, as super-continuum generation [28].

The considered problem has a key difficulty: even if the geometrical localization of the wave-function occurs in a reduced dimensionality (e.g., a curved surface in a three dimensional space), when nonlinearity is very effective, the wave-function spreads over the three-dimensional (3D) space. We may hence question any treatment based on nonlinear wave equations with reduced dimensionality, and

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have to test theoretical predictions by using 3D simulations.

In this article, we investigate the frustration of geometrical localization due to a shock-wave generated by a extreme defocusing nonlinearity. We study this effect by one-dimensional (1D) theoretical analysis and 3D+1 simulations of nonlinear Schrödinger, or Gross-Pitaevskii (GP), equation. We report the first analysis of DSW in curved potentials.

## 2 The model

The considered adimensionalized three-dimensional nonlinear Schrödinger, or GP, equation is

$$\mathrm{i}\Psi_t = -\nabla^2\Psi + V_{3D}(r)\Psi - \chi|\Psi|^2\Psi. \tag{1}$$

In Eq. (1), $\chi = \pm 1$ determines the sign of the nonlinearity, and $N = \int |\Psi|^2 \mathrm{d}^3\boldsymbol{r}$ is the number of atoms. We consider a curved potential, sketched in Fig. 1,

$$V_{3D} = V_1(q_1) + V_\perp(q_2, q_3), \tag{2}$$

where $q_1 = q$ is the longitudinal coordinate along the arc, and $q_{2,3}$ are the transverse coordinates [20]. $V_\perp$ determines the transverse confinement along an arbitrary curve, whose curvilinear coordinate is given by $q$. $V_1(q_1)$ is a weak longitudinal trapping potential, whose effect becomes negligible for strong curvature, as detailed below. Specifically, we have a parabolic potential

$$V_\perp(q_2, q_3) = w\left(q_2^2 + q_3^2\right), \tag{3}$$

$$V_1(q_1) = w_1 q_1^2, \tag{4}$$

with $w \gg w_1$. Being $\lambda^2 = w/w_1$, the 1D reduction holds true as $\lambda^2 \to \infty$ [20], letting

$$\Psi(q_1, q_2, q_3) = l_\perp \psi_\perp(q_2, q_3)\psi(q_1)\exp(-\mathrm{i}E_\perp t), \tag{5}$$

with $\int |\psi_\perp|^2(q_2, q_3)\mathrm{d}q_2\mathrm{d}q_3 = 1$, and $E_\mathrm{T}$ the transverse eigenvalue. The transverse localization length is

$$l_\mathrm{T}^{-2} = \int |\psi_\perp(q_2, q_3)|^4 \mathrm{d}q_2\mathrm{d}q_3, \tag{6}$$

and the normalization condition holds

$$\int |\psi(q)|^2 \mathrm{d}q = N l_\perp^{-2} \equiv P. \tag{7}$$

The linear ground state of the parabolic potential is $\psi_\perp = \sqrt{2/\pi l_\mathrm{T}^2}\exp\left(-q_2^2/l_\mathrm{T}^2 - q_3^2/l_\mathrm{T}^2\right)$ with $l_\mathrm{T} = (4/w)^{1/4}$, and $E_\mathrm{T} = 2\sqrt{w}$. The resulting 1D GP equation with geometrical potential $V_\mathrm{G}$ is given by

$$\mathrm{i}\partial_t\psi = -\partial_q^2\psi + V(q)\psi - \chi|\psi|^2\psi, \tag{8}$$

where $V = V_1 + V_\mathrm{G}$, and $V_\mathrm{G}$ is expressed in terms of the local curvature $K(q)$:

$$V_\mathrm{G}(q) = -\frac{K^2(q)}{4}. \tag{9}$$

$V_\mathrm{G}(q)$ is a an effective potential that arise from the 1D reduction of the original 3D Eq. (1) [20]. The bound state is determined by

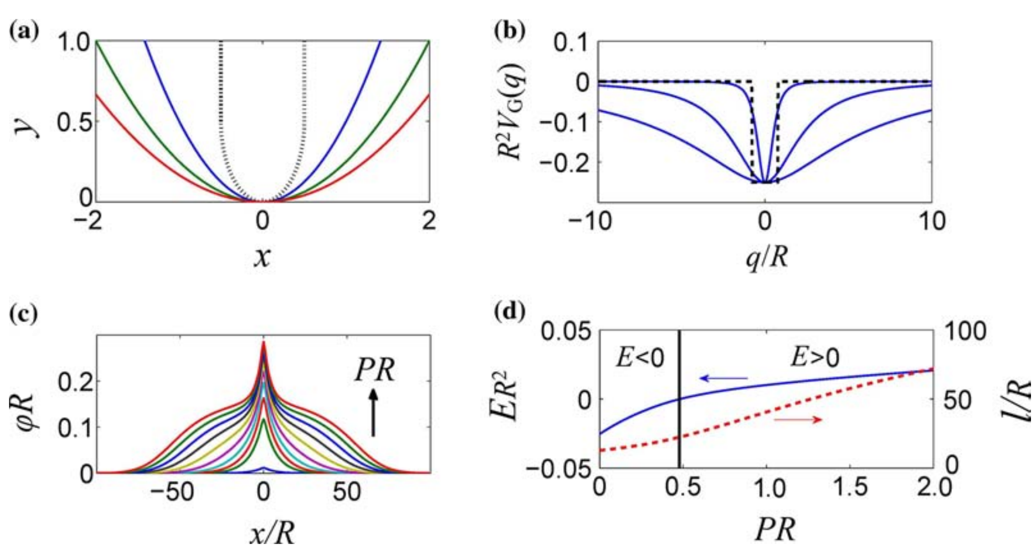


**Fig. 1** (Color online) **a** Potential profile for the piecewise case (dashed, $R = 0.5$) and for the parabolic cases ($y = kx^2$ with $R = 1/2k = 1, 2, 3$); **b** geometrical potential $V_G$ for the cases in panel (**a**); **c** shape of the bound state for the parabolic potential for various powers ($PR$ goes from $10^{-3}$ to 2.6); **d** nonlinear eigenvalue versus power (left axis), and localization length (right axis), $w_1 R^2 = 6.25 \times 10^{-6}$