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Long range topological correlations in cellular patterns

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ABSTRACT

In random 2D cellular assemblies, maximum entropy inference yields a specific form for the topological pair correlation, bi-affine in the cell charges (6-polygonality). By a self-consistent method, we show that the long distance behaviour of the pair correlation function is related to sum rules involving moments of the quadratic coefficients of the bi-affine form. The correlation function is predicted to decay like the 4th power of inverse distance if the first moment does not vanish, faster otherwise. The lowest sum rule expresses the screening of topological charge in the foam. Comparison with sparse available experimental data is not conclusive.

1. Introduction

Foams are random but finding their probability distribution is still an intriguing challenge. Even if all observables are prone to follow some distribution laws, the present study is devoted to elementary topological aspects such as counting cell sides, neighbourhood, stepwise distance, with main focus on two-body correlations. Space-filling and randomness are essential, common characteristics of foams; the question addressed here is what are the implications of those properties alone in the structure and rigidity of foams, independently of the geometrical and energetic details specific to each sample of the tremendous variety of physico-chemical systems forming cellular aggregates. So far, only a few investigations have been devoted to correlations beyond nearest neighbours. Pair correlations at arbitrary distance were analysed in [1,2,3,4,5], and measured in [4].

The inevitable constraint of space-filling conditions most of the topology. Disorder is treated by Maximum Entropy, one of the few methods able to predict some aspects of the probability distributions [6]. Maximum entropy (maxent) arguments yield a specific form for the pair correlations at arbitrary distance [2].

The present analysis is devoted to bi-dimensional foams, subject to extensive theoretical and experimental investigations [1–4,6–25].

In [5], we showed that the asymptotic behaviour of layer populations was controlled by sum rules for the maxent coefficients of the pair correlator. But a closer examination of the asymptotic conditions, in a self consistent way, shows that these sum rules also constrain the decay of the pair correlation: either as j^{-4} if the first moment $M_1 \neq 0$, or faster if $M_1 = 0$, as a function of distance j.

2. Foam statistics

For consistency, we briefly recall some elements of foam topology and statistics. Details can be found in [5].

A foam $\mathcal{F}$ divides space into $N = |\mathcal{F}|$ polygonal cells. Here $\mathcal{F}$ is viewed as a set of cells and $|\mathcal{F}|$ represents the number of elements in the set.

2.1. One cell statistics

The local observable is n, the number of sides of each cell ("polygonality"), or the *topological charge* $q = 6 - n$. The fraction of $(6 - q)$-sided cells is $p(q) = N(q)/N$.

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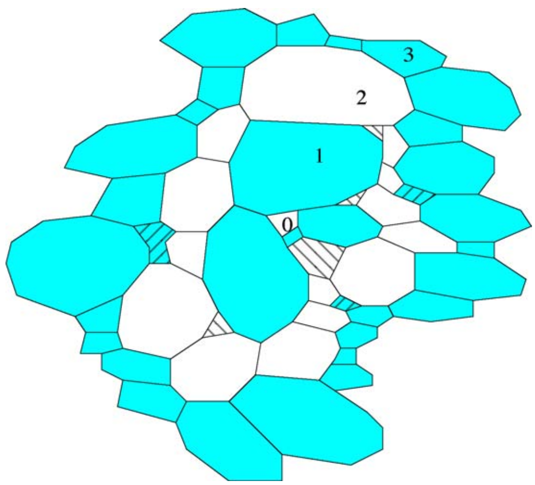

Fig. 1. Stratification into layers around a cell 0. The defects are marked hatched.

The laws of Plateau, vertex coordination (or degree) $z = 3$, and Euler imply

$$\langle q\rangle = \langle 6-n\rangle = \sum_{q<6} qp(q) \to 0\, as\, N \to \infty. \quad (1)$$

under mild assumptions on the foam boundary.

The second moment is $\mu_2 = \langle q^2\rangle = \langle (n-6)^2\rangle$.

2.2. Pair of cells, correlations

The topological distance j between two cells is the minimal number of steps needed to join the cells, where a step is a move from a cell to a contiguous neighbour cell. For example, in Fig. 1, cells 0 and 2 form a $(2, -5)$ pair at distance $j = 2$. The j th *layer* around a given cell o, lay $(j|o)$, is the set of cells at distance j from o (Fig. 1). It has population $K_j(o) = |\text{lay}\,(j|o)|$. The average over $(6-q)$-sided central cells is $\langle K_j(q)\rangle$ and the overall average is $\langle\langle K_j\rangle\rangle$.

The joint distribution $p_j^{(2)}(q_1, q_2)$ — probability that a (q_1, q_2) -pair of cells occurs at mutual distance j – and the corresponding marginal distribution $s_j(q) = \sum_{q'} p_j^{(2)}(q, q')$ — probability that a cell is at distance j from a $(6-q)$ -sided one — satisfy $s_j(q) = \frac{\langle K_j(q)\rangle}{\langle\langle K_j\rangle\rangle} p(q)$.

The correlator $A_j(q_1, q_2)$ and correlation function $g_j(q_1, q_2)$ are defined by

$$p_j^{(2)}(q_1, q_2) = A_j(q_1, q_2)\frac{p(q_1)p(q_2)}{\langle\langle K_j\rangle\rangle} \quad (2)$$

$$p_j^{(2)}(q_1, q_2) = g_j(q_1, q_2)s_j(q_1)s_j(q_2). \quad (3)$$

Both account for the statistical dependence of the simultaneous occurrence of a (q_1, q_2) -pair of cells at distance j. They only differ by the way they are normalised: the correlation function is 1 whereas the correlator is $\langle\langle K_j\rangle\rangle$ in independent situations.

The following consistency (or tautological) identities were proved in [1,2], involving the average population and charge of layer j:

$$\sum_{q'} p(q')A_j(q', q) = \langle K_j(q)\rangle \quad (4)$$

$$\sum_{q'} q'p(q')A_j(q', q) = \langle Q(\text{lay}(j|q))\rangle \quad (5)$$

Averages are conditioned by the central cell having $(6-q)$ sides. The charge Q of a set of cells —here, a layer— is the sum of the individual charges.

Remark. The "average charge per cell" in j -layers, $\prec q_j(q)\succ \equiv \langle Q(\text{lay}(j|q))\rangle / \langle K_j(q)\rangle$, appears in the Aboav–Weaire law (for $j = 1$).

3. Recurrence equation

3.1. Recursion equation

The central equation was derived in (1) and (2):

$$\Delta\langle K_j(q)\rangle + \langle Q(\text{lay}(j|q))\rangle = \langle I_j(q)\rangle \simeq 0, \quad (6)$$

where $\Delta K_j = K_{j+1} - 2K_j + K_{j-1}$ is discrete Laplacian and $\langle I_j(q)\rangle$ is a contribution due to defects, the cells of layer j which have no edge in common with the next layer, $j + 1$ (Fig. 1). It is assumed that, on average, this contribution vanishes: $\langle I_j(n)\rangle = 0$.

3.2. Maximum entropy

Extending a bi-affine formula first introduced for nearest neighbours ($j = 1$) [26], maximum entropy arguments (maxent) and the recursion relation (6) give the following expressions [2]

$$A_j(q_1, q_2) = b_j - a_j(q_1 + q_2) + \sigma_j q_1 q_2, \quad (7)$$

$$\langle K_j(q)\rangle = b_j - a_j q, \quad (8)$$

where σ_j, a_j, b_j are real parameters for $j = 1, 2, \ldots$. In the infinite foam limit, the average of (8) gives $\langle\langle K_j\rangle\rangle = b_j$.

From (2), (3), (7) and (8), one deduces the correlation function

$$g_j(q_1, q_2) - 1 = \frac{A_j(q_1, q_2)\langle\langle K_j\rangle\rangle}{\langle K_j(q_1)\rangle\langle K_j(q_2)\rangle} - 1 = \frac{\frac{\sigma_j}{b_j} - \left(\frac{a_j}{b_j}\right)^2}{(1 - \frac{a_j}{b_j}q_1)(1 - \frac{a_j}{b_j}q_2)} q_1 q_2. \quad (9)$$

3.3. Asymptotic freedom

In normal systems of statistical physics, distant events become uncorrelated. In foams, this was first measured by [4].

$$As\, j \to \infty,\ g_j(q_1, q_2) - 1 \to 0. \quad (10)$$

In terms of the maxent coefficients (7), (8), using (9), the decay of $g_j - 1$ amounts to

$$\frac{\sigma_j}{b_j} - \left(\frac{a_j}{b_j}\right)^2 \to 0. \quad (11)$$

In fact, following [5], we assume slightly stronger conditions:

$$\frac{a_j}{b_j} \to 0, \quad (12)$$

$$|\sigma_j| = O(j^{-2-\alpha}) \quad \text{for some}\, \alpha > 0. \quad (13)$$

These conditions imply (11) because $b_j \gtrsim j$ as $j \to \infty$ (Section 4.1).

The limit (12) expresses the fact that $\langle K_j(q)\rangle \to \langle\langle K_j\rangle\rangle$ and $s_j(q) \to p(q)$: conditioning by the central cell charge (for $\langle K_j(q)\rangle$, see Section 2.2) or presence (for $s_j(q)$) has a vanishing effect at large distance.

For (13), we will see, in Section 4.2, that $\{\sigma_j\}$ is related to the average charge excess induced by conditioning on a central charge q. So, again, (13) expresses the asymptotic decay of influence.