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Colloids and Surfaces A: Physicochem. Eng. Aspects 323 (2008) 50–55

COLLOIDS AND SURFACES A

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A mathematical model for the evaporation of a thin sessile liquid droplet: Comparison between experiment and theory

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Received 22 June 2007; received in revised form 27 August 2007; accepted 18 September 2007
Available online 25 September 2007

Abstract

A mathematical model for the quasi-steady diffusion-limited evaporation of a thin axisymmetric sessile droplet of liquid with a pinned contact line is formulated and solved. The model generalises the theoretical model proposed by Deegan et al. [Contact line deposits in an evaporating drop, Phys. Rev. E, 62 (2000) 756–765] to include the effect of evaporative cooling on the saturation concentration of vapour at the free surface of the droplet, and the dependence of the coefficient of diffusion of vapour in the atmosphere on the atmospheric pressure. The predictions of the model are in good qualitative, and in some cases also quantitative, agreement with recent experimental results. In particular, they capture the experimentally observed dependence of the total evaporation rate on the thermal conductivities of the liquid and the substrate, and on the atmospheric pressure.

Keywords: Evaporation; Liquid droplet; Evaporative cooling; Atmospheric pressure

1. Introduction

The evaporation of liquid droplets is of fundamental importance in a huge variety of practical situations ranging from technological applications such as ink-jet printing, spray cooling and various coating processes, to a variety of biological and geophysical contexts. As a result, droplet evaporation has been the subject of considerable theoretical and experimental research in recent years. Significant recent papers include those by Deegan [1], Deegan et al. [2], Hu and Larson [3–5], Popov [6], Poulard et al. [7], Sultan et al. [8], Shahidzadeh-Bonn et al. [9], and Girard et al. [10].

Physical experiments conducted recently by David et al. [11] using a variety of liquids and substrates show that the thermal conductivities of the liquid and the substrate, and the atmospheric pressure, can have a significant effect on the total evaporation rate. Neither of these effects is captured by the widely used pioneering theoretical model proposed by Deegan et al. [2] (hereafter referred to simply as "the Deegan model" for brevity).

In this paper, a mathematical model for the quasi-steady diffusion-limited evaporation of a thin axisymmetric sessile droplet of liquid with a pinned contact line is formulated and solved. This model generalises the Deegan model to include the effect of evaporative cooling on the saturation concentration of vapour at the free surface of the droplet, and the dependence of the coefficient of diffusion of vapour in the atmosphere on the atmospheric pressure. For simplicity, the present initial model is, however, restricted to the special case of thin droplets with small contact angles. The results presented here show that the predictions of the model are in good qualitative, and in some cases also quantitative, agreement with the experimental results. In particular, they capture the experimentally observed dependence of the total evaporation rate on the thermal conductivities of the liquid and the substrate, and on the atmospheric pressure.

The present paper describes some aspects of the work presented at the 3rd International Workshop on Bubble and Drop

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0927-7757/$ – see front matter
doi:10.1016/j.colsurfa.2007.09.031

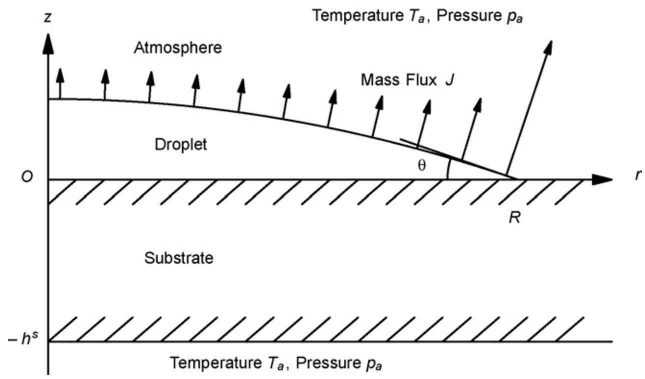

Fig. 1. Geometry of the problem.

Interfaces held on 25–28 March 2007 in Granada, Spain. A preliminary account of part of this work was given by Dunn et al. [12].

2. The mathematical model

Consider the quasi-steady diffusion-limited evaporation of a thin axisymmetric sessile droplet of liquid with constant density ρ, surface tension σ, and thermal conductivity k on a thin horizontal substrate of constant thickness $h^{\rm s}$ with constant thermal conductivity $k^{\rm s}$. Referred to cylindrical polar coordinates (r, ϕ, z) with origin on the substrate at the centre of the droplet and with the z axis vertically upwards, the shape of the free surface of the droplet is denoted by $z = h(r, t)$, the upper surface of the substrate by $z = 0$, and the lower surface of the substrate by $z = -h^{\rm s}$, as shown in Fig. 1.

The contact lines of evaporating droplets are typically pinned by surface roughness (or other) effects during the first stage of their evaporation, but can de-pin prior to complete evaporation. All of the experimental results for temperature and evaporation rate reported by David et al. [11] are for droplets in this first (pinned) stage, and so in the present model we assume that the droplet radius R remains constant. We also assume that the droplet is sufficiently small that surface tension effects dominate gravitational effects, and that the droplet is sufficiently thin (in particular, that the contact angle $\theta = \theta(t)$ is sufficiently small) that it has the simple quasi-static parabolic shape

$$h = \frac{\theta(R^2 - r^2)}{2R} \tag{1}$$

with volume $V = V(t)$ given by

$$V = \frac{\pi R^3 \theta}{4}. \tag{2}$$

While the former assumption is well justified for the experiments of David et al. [11], the latter assumption is more questionable (see the discussion about this assumption in Section 4). The total evaporation rate is given by

$$-\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{2\pi}{\rho}\int_0^R J(r, t)\, r\, \mathrm{d}r, \tag{3}$$

where $J = J(r, t)$ (>0) is the local evaporative mass flux from the droplet.

The atmosphere surrounding the droplet and the substrate is assumed to be at constant atmospheric temperature $T_{\rm a}$ and pressure $p_{\rm a}$. Since both the droplet and the substrate are thin, their temperatures, denoted by $T = T(r, z, t)$ and $T^{\rm s} = T^{\rm s}(r, z, t)$, satisfy

$$\frac{\partial^2 T}{\partial z^2} = 0, \quad \frac{\partial^2 T^{\rm s}}{\partial z^2} = 0. \tag{4}$$

The mass flux from the droplet satisfies the local energy balance

$$LJ = -k\frac{\partial T}{\partial z} \tag{5}$$

on $z = h$ for $r < R$, where L is the latent heat of vaporisation. Hence, assuming that both the temperature and the heat flux are continuous between the droplet and the wetted part of the substrate, and that the lower surface of the substrate is at the atmospheric temperature $T_{\rm a}$, we have

$$T = T_{\rm a} - LJ\left(\frac{z}{k} + \frac{h^{\rm s}}{k^{\rm s}}\right), \quad T^{\rm s} = T_{\rm a} - \frac{LJ}{k^{\rm s}}(z + h^{\rm s}), \tag{6}$$

showing clearly the evaporative cooling of both the droplet and the substrate.

Assuming that transport of vapour in the atmosphere is dominated by diffusion (see, for example, Popov [6]), the concentration of vapour in the atmosphere above the droplet and the substrate, denoted by $c = c(r, z, t)$, satisfies Laplace's equation,

$$\nabla^2 c = 0. \tag{7}$$

Since the droplet is thin, Eq. (7) holds in the half-space $z > 0$, and the boundary conditions for c on the free surface of the droplet may be imposed on $z = 0$ rather than on $z = h$.

At the free surface of the droplet, we assume that the atmosphere is saturated with vapour so that $c = c_{\rm sat}(T)$, where the saturation concentration $c_{\rm sat} = c_{\rm sat}(T)$ is assumed to be a linearly increasing function of temperature given by

$$c_{\rm sat}(T) = c_{\rm sat}(T_{\rm a}) + \left.\frac{\mathrm{d}c_{\rm sat}}{\mathrm{d}T}\right|_{T=T_{\rm a}}(T - T_{\rm a}) \tag{8}$$

on $z = 0$ (rather than on $z = h$) for $r < R$. On the dry part of the substrate there is no mass flux, i.e.

$$\frac{\partial c}{\partial z} = 0 \tag{9}$$

on $z = 0$ for $r > R$, and far from the droplet the concentration of vapour approaches its ambient value, i.e.

$$c \to Hc_{\rm sat}(T_{\rm a}) \tag{10}$$

as $(r^2 + z^2)^{1/2} \to \infty$, where H is the relative saturation of the atmosphere far from the droplet (which may be zero or nonzero). Once c is known the mass flux from the droplet is given by

$$J = -D\frac{\partial c}{\partial z} \tag{11}$$