Journal of Colloid and Interface Science 417 (2014) 37–50

Contents lists available at ScienceDirect

Journal of Colloid and Interface Science

www.elsevier.com/locate/jcis

Theory of axisymmetric pendular rings

Boris Y. Rubinstein [a], Leonid G. Fel [b,*]

[a] Stowers Institute for Medical Research, 1000 E 50th St., Kansas City, MO 64110, USA
[b] Department of Civil and Environmental Engineering, Technion –Israel Institute of Technology, Haifa 32000, Israel

ARTICLE INFO

Article history:
Received 11 August 2013
Accepted 13 November 2013
Available online 23 November 2013

The paper is dedicated to the memory of our friend and bright scientist A. Golovin (1962–2008)

Keywords:
Plateau problem
Young-Laplace equation
Axisymmetric pendular rings

ABSTRACT

We present the theory of liquid bridges between two solids, sphere and plane, with prescribed contact angles. We give explicit expressions for curvature, volume and surface area of pendular ring as functions of the filling angle ψ for all available types of menisci: catenoid, sphere, cylinder, nodoid and unduloid (the meridional profile of the latter may have inflection points). There exists a rich set of solutions of the Young-Laplace equation for the shape of an axisymmetric meniscus of constant mean curvature. In case when the solids do not contact each other, these solutions extend Plateau's sequence of meniscus evolution observed with increase of the liquid volume to include the unduloids at small filling angle, unduloids with multiple inflection points and multiple catenoids.

The Young-Laplace equation with boundary conditions can be viewed as a nonlinear eigenvalue problem. Its unduloid solutions, menisci shapes and curvatures $H_n^s(\psi)$, exhibit a discrete spectrum and are enumerated by two indices: the number n of inflection points on the meniscus meridional profile $\mathcal{M}$ and the convexity index $s=\pm 1$ determined by the shape of a segment of $\mathcal{M}$ contacting the solid sphere: the shape is either convex, $s=1$, or concave, $s=-1$.

For the fixed contact angles the set of the functions $H_n^s(\psi)$ behaves in such a way that in the plane $\{\psi, H\}$ there exists a bounded domain where $H_n^s(\psi)$ do not exist for any distance between solids. The curves $H_n^s(\psi)$ may be tangent to the boundary of domain which is a smooth closed curve. This topological representation allows to classify possible curves and introduce a saddle point notion. We observe several types of saddle points, and give their classification.

1. Introduction

The problem of pendular ring (PR) arises when a fluid forms an axisymmetric liquid bridge with interface (meniscus) between two axisymmetric solids. This problem includes a computation of liquid volume V, surface area S and surface curvature H and was one of gems in mathematical physics of the 19th century. In the last decade the PR problem became again an area of active research due to the stability problem of the PR shapes [4] and its importance has grown for various applications in soil engineering and porous media [2,1], in medicine [10] and plant biology [9].

In 1841 Delaunay [3] classified all nontrivial surfaces of revolution with constant mean curvature (CMC) in $\mathbb{R}^3$ by solving the Young-Laplace (YL) equation and showed that they are obtained by tracing a focus of a conic section rolled on a line, and revolving the resulting curve around the axis of symmetry. These are *cylinder* (Cyl), *sphere* (Sph), *catenoid* (Cat), *nodoid* (Nod) and *unduloid* (Und). The two last of them are defined through the elliptic integrals and may appear of two kinds, *concave* (-) and *convex* (+), depending on constant sign of the meridional profile $\mathcal{M}$ curvature. Two more types of menisci, *inflectional unduloid* and *nodoid*, appear when meridional section $\mathcal{M}$ curvature changes its sign along the meniscus.

In 1864 Plateau [12] applied this classification to analyze the figures of equilibrium of a liquid mass, and was the first who discovered [13] a standard sequence of meniscus evolution observed with increase of the liquid volume. According to [11] the Plateau sequence reads:

$$\mathrm{Nod}^- \to \mathrm{Cat} \to \mathrm{Und}_0^- \to \mathrm{Und}_1^- \to \mathrm{Und}_0^+ \to \mathrm{Sph} \to \mathrm{Nod}^+, \tag{1}$$

where subscripts denote the number of inflection points on the meniscus meridional section $\mathcal{M}$. Even so, solution of the PR problem leads to the eigenvalue problem for curvature H that requires computation of elliptic integrals and was not available before the computer era. A complete review on different methods used to find actual solutions of YL equation or the equivalent variational problem (see Howe's dissertation [6]) throughout the last century can be found in [11].

The PR problem can be considered in two different setups: with fixed contact lines (CL) [8] and with free CL [11]. Their differ in boundary conditions. In the first case the fluid CL is pinned and

* Corresponding author. Fax: +972 48228898.
E-mail addresses: bru@stowers.org (B.Y. Rubinstein), lfel@technion.ac.il (L.G. Fel).

0021-9797/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jcis.2013.11.038

the contact angle at the solid surfaces varies as, for example, the volume of liquid in the bridge is varied. In the second case the contact angles at the solid surfaces are specified, and the location (filling angle) of CL changes with the volume.

In 1975 Orr, Scriven and Rivas [11] derived formulas for V, S and H for various menisci in the case of solid sphere of radius R above the solid plane for $D \geqslant 0$ with contact angles θ_1 and θ_2 on sphere and plane, respectively; D is a distance between the sphere and the plane. In the case of touching solids numerical computations verified the Plateau sequence (1) when the liquid volume is increasing.

During the past decades the work [11] became classical, albeit throughout a vast number of references (more than 350 to date) no attempt was made to extend this explicit analysis. Formulas in [11] for the separated solids were also left without detailed analysis. There is a substantial difference in behavior of the function $H(\psi, d)$ for $d = 0$ and $d > 0$ where $d = D/R$. This can be seen in Fig. 1 where we consider for small filling angles ψ two types of menisci between touching (a) and nontouching (b) sphere and plane having the same set of contact angles $\theta_1 < \pi/2$ and $\theta_2 = \pi/2$ (see Fig. 2).

The sphere-plane geometry approaches its *wedge* (a) and *slab* (b) limits. Estimate two principal radii, meridional R_v and horizontal R_h, in both cases for $\psi \ll 1$. In the case (a) they are of different signs, $R_v < 0, R_h > 0$. A simple trigonometry gives

$$2R_v/R \simeq -\psi^2/\cos\theta_1, \quad R_h/R \simeq \psi, \quad RH \simeq -\psi^{-2}\cos\theta_1.$$

The dependence $H \simeq -\psi^{-2}, \psi \ll 1$, describes the asymptotics $H(\psi)$ of the Nod^- meniscus for two touching solids found in [11]. In the second case (b) we have,

$$R_v/D \simeq -1/\cos\theta_1, \quad R_h/R \simeq \psi, \quad 2RH \simeq \psi^{-1},$$

where another asymptotics holds, $H \simeq \psi^{-1}$. It changes the whole sequence (1) giving rise to the Cat menisci for two different filling angles ψ, or to a single degenerated Cat meniscus, or to disappearance of both of them.

In this paper we present a theory of pendular rings between two solids, sphere and plane, with free CL and general setup, when the solids are separated, touching or intersecting each other. We derive the expressions for curvature $H(\psi)$, volume $V(\psi)$ and surface area $S(\psi)$ for all available menisci type including those omitted in [11].

The paper is organized in eight sections and appendix. In Section 2 we give the problem setup and solve the YL equation through the elliptic integrals. We find explicit expressions for the shape, $V(\psi)$, $S(\psi)$ and $H(\psi)$. We introduce a new function $\alpha(\psi)$ related to the curvature $H(\psi)$ which becomes a main tool of the PR evolution analysis.

Section 3 discusses general curvature behavior for different types of menisci. We consider the existence of catenoids in the menisci sequence for nontouching solids. In contrast to the case $d = 0$ [11], when the Nod^- menisci exist for small ψ, in the case $d > 0$ the Und_n^- menisci come first. This allows the existence of two catenoids in the menisci sequence or their annihilation. We show that $\alpha(\psi)$ is bounded for some types of the menisci. When these bounds are crossed, the corresponding menisci transform one into another; these transitions, their sequence and smoothness are discussed in Section 4.

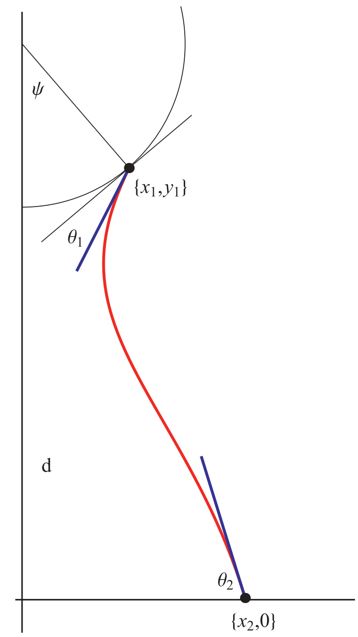

Fig. 2. A sketch of meniscus between plane and sphere showing the contact angles θ_1, θ_2, filling angle ψ and coordinates of the endpoints.

In Section 5 we develop a topological approach to study a set of curves $\alpha_n(\psi)$ enumerated by the number n of inflection points on $\mathcal{M}$. These curves in the plane $\{\psi, \alpha\}$ are confined within domain $\Delta' = \{0 \leqslant \psi \leqslant \pi - \theta_1,\ 0 \leqslant \alpha \leq 1\}$ with subdomain $\mathcal{B} \subset \Delta'$ prohibited for $\alpha_n(\psi)$ to pass through. The curves $\alpha_n(\psi)$ are tangent to the $\mathcal{B}$'s boundary which is a smooth closed curve. This representation allows to classify all curves $\alpha_n(\psi)$ and introduce a notion of saddle point. We observe several types of saddle points, their classification is presented in Appendix. In Section 6 we analyze the menisci evolution in the cases of touching and intersecting solid bodies which is essentially different from the case of separated bodies. The last section is devoted to discussion of the open problems of PR.

2. Young-Laplace equation and its solutions

The PR problem can be posed as follows: find the CMC surface of revolution characterized by a curvature $\widetilde{H}$ satisfying the YL equation valid in case of negligibly small gravity effect

$$2\widetilde{H} = \frac{z''}{(1 + z'^2)^{3/2}} + \frac{z'}{r(1 + z'^2)^{1/2}}, \tag{2}$$

where $z(r)$ and r are cylindrical coordinates of the meniscus. Introduce new variables $x = r/R$ and $y = z/R$ for which we have $dy/dx = \tan t$, where t is an angle between the normal to the meniscus surface and the vertical axis. Introduce another parameter u and rewrite this equation for nondimensional curvature $H = R\widetilde{H}$,

$$2H = du/dx + u/x, \quad u = y'/\sqrt{1 + y'^2} = \sin t, \quad y' = \tan t. \tag{3}$$

The contact angles with the solid bodies are θ_1 (with the sphere) and θ_2 (with the plane). The boundary conditions read

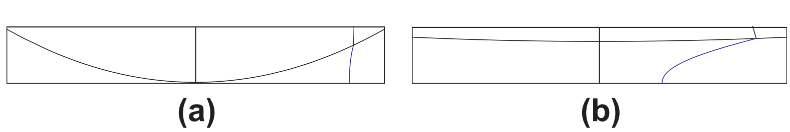

Fig. 1. (a) The shape of meniscus for $\theta_1 = \pi/6$, $\theta_2 = \pi/2$, $\psi = \pi/30$, with (a) $d = 0$, $HR = -69.57$, and (b) $d = 0.076$, $HR = 2.14$.