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Dimensionless scaling methods for capillary rise

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ABSTRACT

In this article the different dimensionless scaling methods for capillary rise of liquids in a tube or a porous medium are discussed. A systematic approach is taken, and the possible options are derived by means of the Buckingham π theorem. It is found that three forces (inertial, viscous and hydrostatic forces) can be used to obtain three different scaling sets, each consisting of two dimensionless variables and one dimensionless basic parameter. From a general point of view the three scaling options are all equivalent and valid for describing the problem of capillary rise. Contrary to this we find that for certain cases (depending on the time scale and the dominant forces) one of the options can be favorable. Individually the different scalings have been discussed and used in literature previously, however, we intend to discuss the three different sets systematically in a single paper and try to evaluate when which scaling is most useful. Furthermore we investigate previous analytic solutions and determine their ranges of applicability when compared to numerical solutions of the differential equation of motion (momentum balance).

1. Introduction

There are numerous applications of capillary transport phenomena ranging from daily life (writing with ink) to complex engineering applications (fluid management in space) and pure academic interest (validation of CFD tools). Thus there are many publications dealing with this problem, its mathematical description and its physical explanation [1–7]. To obtain a better understanding of a problem its dimensionless consideration is always of interest. Here the Buckingham π theorem [8] can be used to obtain appropriate dimensionless scalings. In literature there are several papers applying dimensionless numbers to the problem of capillary rise. Ichikawa and Satoda [9] focus on experiments with horizontal capillaries, Dreyer et al. [10] and Stange et al. [11,12] on capillaries in a microgravity environment. There also exist studies involving gravity, thus leading to different scaling approaches e.g. by Quéré et al. [13,14], Marmur and Cohen [15], Zhmud et al. [16], Lee and Lee [17] or Fries and Dreyer [18,19]. McKinley [20] investigates dimensionless groups for free surface flows with a focus on complex fluids. In this paper we now intend to follow a systematic approach to dimensionless scaling of capillary rise, and to compare the different derived options.

The basis for the dimensionless scalings is the differential equation of motion of the liquid inside a capillary tube. It can be derived by solving an integral balance of the linear momentum in an appropriate control volume [5]. To solve the integrals and to obtain the boundary conditions some assumptions have been made. First of all the viscous losses in the tube are described using the Hagen-Poiseuille law. Also the capillary pressure is assumed to be constant, hence a static contact angle θ is used (e.g. see [6,19]). Furthermore entry effects and losses in the liquid reservoir are neglected. With these assumptions the equation of motion is given by (e.g. [3,16])

$$-\rho\frac{d(h\dot{h})}{dt} = -\frac{2\sigma\cos\theta}{R} + \frac{8\mu h}{R^2}\dot{h} + \rho g h \quad (\text{for } \dot{h} > 0). \tag{1}$$

In this equation the momentum change (inertia, left hand side) is balanced by the capillary pressure, the viscous forces and the hydrostatic pressure (left to right). σ refers to the surface tension, R to the inner tube radius, ρ to the fluid density, g to gravity and μ to the fluid viscosity. It is interesting to note that Eq. (1) is only valid for a rising column. For a falling column – as it occurs in oscillating cases – the different flow characteristics at the tube inlet have to be considered. While for the rising column it acts as a sink, a jet is emitted for the falling column. For the descending case, a $\dot{h}^2$ term included in the left hand side of Eq. (1) has to be omitted to obtain

$$-\rho h\ddot{h} = -\frac{2\sigma\cos\theta}{R} + \frac{8\mu h}{R^2}\dot{h} + \rho g h \quad (\text{for } \dot{h} < 0), \tag{2}$$

as shown by Lorenceau et al. [21].

The momentum balance can also be given for the capillary rise of liquids in porous media, here the viscous term is replaced by the Darcy law

$$-\rho\frac{d(h\dot{h})}{dt} = -\frac{2\sigma\cos\theta}{R} + \frac{\phi\mu h}{K}\dot{h} + \rho g h \quad (\text{for } \dot{h} > 0). \tag{3}$$

ϕ denotes the porosity of the structure, and K its permeability.

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2. Dimensionless scaling

In this section the different dimensionless scaling options will be discussed. The Buckingham π theorem and the approach described by White [22] is used. The relevant definitions shall be introduced briefly:

- Dimensional variables are the basic output of the experiment, and normally the ones to be shown in a diagram. They vary during a given run. In our case h and t (see Fig. 1).
- Dimensional parameters affect the variables, and may vary from case to case, however remain constant during a given run. In our case a, b and c, see Eqs. (4)–(6) below.
- Fundamental units are the units of the variables and parameters e.g. meter, kilogram, second.
- Scaling parameters are chosen to convert the variables to a dimensionless form. In our case: two can be chosen.
- Basic parameter is the – in our case one – remaining parameter.
- Dimensionless variables are the variables made dimensionless by the scaling parameters.
- Dimensionless basic parameter is the basic parameter made dimensionless using the scaling parameters.

In a graphic representation of the dimensionless solution the axes are the dimensionless variables, while the dimensionless basic parameter is varied to plot a set of curves [22] (e.g. Fig. 2). With varying dimensionless basic parameter the influence of the basic parameter (and the corresponding force) can be observed. Regarding Eqs. (1) and (3) we may define the following dimensional parameters:

$$a = \frac{\rho R}{2\sigma\cos\theta}, \tag{4}$$

$$b = \frac{4\mu}{R\sigma\cos\theta} \mathrel{\hat{=}} \frac{\phi\mu R}{2K\sigma\cos\theta}, \tag{5}$$

$$c = \frac{\rho g R}{2\sigma\cos\theta}. \tag{6}$$

For b both the capillary tube and the Darcy version is given. However, in favor of readability, we will not continue to explicate the Darcy version in the further text. Please note that the parameters a, b and c are not identical to those applied in [18,19]. Using the introduced dimensional parameters one can rearrange Eqs. (1) and (3) to obtain

$$\underbrace{a\frac{d(h\dot{h})}{dt}}_{\text{inertial}} + \underbrace{bh\dot{h}}_{\text{viscous}} + \underbrace{ch}_{\text{hydrostatic}} = 1. \tag{7}$$

It can now be observed that the momentum balance has become much more clearly arranged and that each dimensional parameter stands for a single term: a – inertia, b – viscous effects and c – hydrostatic effects. Table 1 summarizes the three different scaling options that will be examined one by one in the next sections.

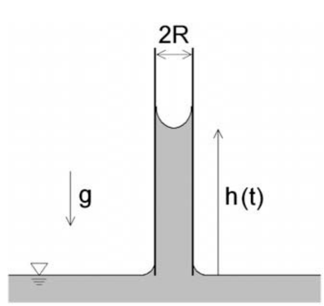

Fig. 1. Liquid rise in a capillary tube [19].

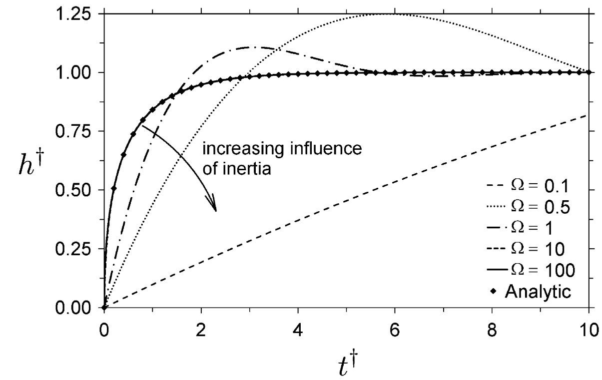

Fig. 2. Plot showing the dimensionless numerical solution of Eqs. (11) and (12). Viscosity and gravity are the scaling forces, inertia is the basic parameter for the set of curves. The points refer to the analytic solution for $\Omega \to \infty$ by Washburn.

3. Viscous effects and gravity as scaling forces (†)

Here, b (viscous effects) and c (gravity) are used as scaling parameters, the remaining parameter a (inertia) is used as basic parameter. The resulting dimensionless variables and the dimensionless basic parameter are derived by applying the Buckingham π theorem as shown in Appendix A

$$\pi_1^\dagger = h^\dagger = ch = \frac{\rho g R}{2\sigma\cos\theta}h, \tag{8}$$

and

$$\pi_2^\dagger = t^\dagger = \frac{c^2 t}{b} = \frac{\rho^2 g^2 R^3}{16\mu\sigma\cos\theta}t. \tag{9}$$

These two dimensionless variables have been used by Zhmud et al. [16] and Fries and Dreyer [18]. The dimensionless basic parameter reads as follows:

$$\pi_3^\dagger = \Omega = \sqrt{\frac{b^2}{ac^2}} = \sqrt{\frac{128\sigma\cos\theta\mu^2}{\rho^3 g^2 R^5}}. \tag{10}$$

According to Quéré et al. [14], we denote the basic dimensionless parameter $\pi_3^\dagger$ as Ω. Here, Ω can be used to measure the influence of inertia. In Fig. 2 it can be seen that for decreasing Ω (increasing inertia, see arrow) the oscillations and the overshoot increase. This is consistent with Quéré et al. who find oscillations to occur for $\Omega \leqslant 2$. It is interesting to note that for all three scaling options presented in this article Ω (=π_3) is mathematically the same, however, its meaning changes from scaling to scaling [22]. Thus Ω always reflects the influence of the chosen basic parameter. For example, as will be shown later in further detail, Ω can become infinite in two limits which are physically very different: For a non inertial case (the Washburn limit) with $a = 0$, and for the no gravity case (the Bosanquet limit) with $c = 0$.

Table 1
Scaling options.

Option	Basic parameter	Scaling parameters
1	a (inertia)	b (viscosity) and c (gravity)
2	b (viscosity)	a (inertia) and c (gravity)
3	c (gravity)	a (inertia) and b (viscosity)