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Analytical modeling of capillary flow in tubes of nonuniform cross section

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ABSTRACT

The interface rise for the flow in a capillary with a nonuniform cross section distribution along a straight center axis is investigated analytically in this paper. Starting from the Navier–Stokes equations, we derive a model equation for the time-dependent rise of the capillary interface by using an approximated three-dimensional flow velocity profiles. The derived nonlinear, second-order differential equation can be solved numerically using the Runge–Kutta method. The nonuniformity effect is included in the inertial and viscous terms of the proposed model. The present model is validated by comparing the solutions for a circular cylindrical tube, rectangular cylindrical microchannels, and convergent–divergent and divergent–convergent capillaries. The validated model has been applied to capillaries with parabolic varying wall, sinusoidal wall, and divergent sinusoidal wall. The inertial and viscous effects on the dynamic capillary rise and the equilibrium height are investigated in detail.

1. Introduction

The dynamic capillary phenomena are of considerable importance for a variety of fields and applications, such as flip-chip encapsulation [1–3], microfluidics [4,5], flows in soil [6] and other porous media [7,8]. The first theoretical work on the dynamic capillary rise can be traced back to 1921 when Washburn [9] conducted a theoretical investigation on the penetration of liquids into circular cylindrical capillaries and porous bodies. The study considered the effects of the capillary force and the viscous drag, while the gravity and other effects were neglected. Washburn [9] concluded that the distance traveled by the meniscus was proportional to the square root of a nondimensional time. The pioneering work of Washburn inspired many theoretical, computational, and experimental studies. Brittin [10] modified the Washburn equation to include the inertial force and the gravitational force by assuming that: (a) the forces acting on the liquid in the steady state are the same as those in the transient state; (b) the contact angle is constant; (c) the wetting of the tube is not a rate-determining factor of the liquid motion. The modified Washburn equation [10] is a second-order, nonlinear differential equation, the solution of which can be obtained in the form of a double Dirichlet series. Xiao et al. [11] derived a general formulation by incorporating the early work of Dreyer et al. [12] on the entrance pressure loss effects and Newman's dynamic contact angle model [13].

In addition to the symmetric and uniform cross-sectional geometries used in many studies, some cited above, similar approaches have also been extended to consider irregular geometries. Mason and Morrow [14] investigated the meniscus curvature in cylindrical capillaries with triangular cross section. In a later work, Mason and Morrow [15] examined the effects of the pore shape in porous media and the contact angle on the capillary displacement curvatures in pore throats formed by the surfaces of equal spheres.

The dynamic capillary rise in cylindrical capillaries with irregular cross sections has also been studied. For example, Turian and Kessler [16] analyzed the one-dimensional axial capillary-driven flow in uniform capillaries with general noncircular cross section subjecting to quadrant symmetry constraints. Ichikawa et al. [17] extended the analysis for circular cylindrical tubes to rectangular microchannels. Experimentally, Jong et al. [18] reported results of capillary flow in rectangular microchannels.

To study the physical process of the capillary flow motion in porous media, theoretical analyses have also been developed for the capillary flow in tubes where the cross-sectional shape varies along the tube axis. Sharma and Ross [19] derived an equation to describe the kinetics of liquid penetration into sinusoidally constricted capillaries by neglecting the inertial and gravitational terms. Staples and Shaffer [20] proposed a similar sinusoidal model including the gravitational term. Debdutt et al. [7,8] studied the flow kinetics in porous ceramics by using a sinusoidal capillary wall to approximate the real flow path in a porous medium. Erickson et al. [21] studied the capillary-driven flow in convergent–divergent and divergent–convergent capillary tubes by using finite element numerical simulations. Young [22] derived a nonuniform

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capillary model similar to that of Sharma and Ross [19] and Staples and Shaffer [20]. Young [22] applied the model to the no-gravity capillary flows in Erickson et al. [21] by reducing the convergent and divergent transition sections to step changes.

The models for capillaries of axially varying cross-sectional shapes in Refs. [7,8,19,20,22] considered the nonuniform geometry effect on the viscous terms in the cross-sectional plane. But the likely variation of the axial velocity component in the main stream direction has not been accounted for in the viscous terms. The inertial effects were also ignored in the models. There are industrial applications of capillary flow in complex porous media where the axial flow variation and the inertial effects can be important. For instance, the rigid-capillary-pressing (RCP) technology is used to improve the dewatering efficiency in the paper-making process [25]. The paper industry is one of the major industrial energy users and a large proportion of this energy is used to dry the wet paper web. Large mechanical presses are used for many paper grades to reduce the water content from approximately 3 kg water/kg fiber to about 1.5 kg water/kg fiber with thermal energy being used to further reduce the water content to about 0.05 kg water/kg fiber. In addition, there are many grades of paper, such as bath and facial tissue, which cannot be mechanically pressed to dewater the sheet in order to maintain end-use properties. For these grades, virtually 100% of the water (approximately 3 kg water/kg fiber) is currently removed via evaporation. Removal of some of this water via capillary action has the potential to substantially decrease the thermal energy use needed for drying these grades. Experiments have shown that capillary dewatering using the layered, porous structures can remove approximately 0.5 to 0.7 kg water/kg fiber, which results in 16–20% thermal energy savings. During this dewatering process, the capillary flow passes through a thin porous medium with large change in pore sizes. The porous medium is composed of layers of lamina of pore sizes ranging possibly from 10 μm to 1000 μm. A more physically realistic modeling of the capillary rise will help advance the understanding of the various fluid dynamic mechanisms at work in such devices.

In this paper, the capillary flow in nonuniform cross-sectional capillaries is investigated analytically. The analysis starts from the Navier–Stokes equations and a governing equation is derived by the integration of the axial momentum equation over the liquid volume. The analysis has been developed by assuming a parabolic distribution for the velocity component in the primary flow direction and has considered the secondary flow components based on mass conservation. Compared to the classic Lucas–Washburn equation [9–11] and the existing nonuniform capillary model equations [7,8,19,20,22], the proposed model incorporates the inertial terms and the viscous terms for nonuniform geometries. The derived nonlinear, second-order differential equation is complex and a MatLab code has been developed to solve the equation numerically by using the adaptive Runge–Kutta–Fehlberg method [23]. The present model is first validated by comparing with the solutions of the existing model equations for the circular cylindrical tubes, rectangular cylindrical microchannels [17,18], and convergent–divergent (C–D) and divergent–convergent (D–C) capillaries [21]. The proposed model is then applied to capillaries with parabolic varying wall, simple sinusoidal wall, and divergent sinusoidal wall. The nonuniform geometry effects are investigated in detail. The simulation results, especially those for capillaries with large variations of cross section, show that there are significant effects of nonuniform geometry in the cases studied.

2. Model equation

The proposed model equation is first derived in Section 2.1. In Section 2.2, we discuss the characters of the model equation.

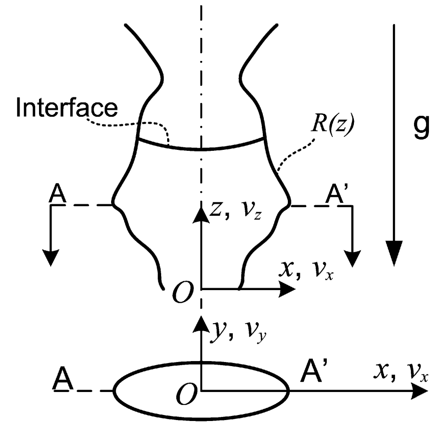

Fig. 1. Sketch of a nonuniform tube with elliptical cross section.

2.1. Derivation

For Newtonian, incompressible fluid flows of constant viscosity, the Navier–Stokes equations can be written in the Cartesian coordinates (x, y, z) as,

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0, \tag{1}$$

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)-\frac{\partial p}{\partial x}, \tag{2}$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)-\frac{\partial p}{\partial y}, \tag{3}$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)-\frac{\partial p}{\partial z}, \tag{4}$$

where v_x, v_y, and v_z are velocity components in x, y, and z directions, respectively. t, ρ, p, and μ denote time, density, pressure, and viscosity, respectively.

The surface tension driven liquid flows in tubes with nonuniform cross section distribution along the axis of the tube are considered. Fig. 1 shows a sketch of such capillary tubes. Elliptic cross section is assumed and the perimeter of the cross section is governed by:

$$x^2+Ky^2=R^2(z) \tag{5}$$

where R denotes the semi-major axis that varies along the z direction. K denotes the square of the ratio of the semi-major axis to the semi-minor axis. The elliptical cross section allows for the modeling of tubes with cross section ranging from circular to near rectangular. In this analysis, the length scale of the local axial variation of the cross-sectional dimension is assumed small compared with the length of the capillary considered, which is valid for most engineering applications.

Due to the small Reynolds number, the capillary flows are laminar in nature and a parabolic profile is assumed for the axial velocity component v_z. That is,

$$v_z(x,y,z,t)=2\frac{dh}{dt}\frac{R^2(h)}{R^2(z)}\left(1-\frac{x^2}{R^2(z)}-K\frac{y^2}{R^2(z)}\right),\quad z<h \tag{6}$$