Available online at www.sciencedirect.com
ScienceDirect
Journal of Colloid and Interface Science 320 (2008) 259–263
Journal of Colloid and Interface Science
www.elsevier.com/locate/jcis

An analytic solution of capillary rise restrained by gravity

N. Fries, M. Dreyer *

Center of Applied Space Technology and Microgravity (ZARM), University of Bremen, Am Fallturm, 28359 Bremen, Germany

Received 1 August 2007; accepted 8 January 2008

Available online 13 January 2008

Abstract

We derive an analytic solution for the capillary rise of liquids in a cylindrical tube or a porous medium in terms of height h as a function of time t. The implicit $t(h)$ solution by Washburn is the basis for these calculations and the Lambert W function is used for its mathematical rearrangement. The original equation is derived out of the 1D momentum conservation equation and features viscous and gravity terms. Thus our $h(t)$ solution, as it includes the gravity term (hydrostatic pressure), enables the calculation of the liquid rise behavior for longer times than the classical Lucas–Washburn equation. Based on the new equation several parameters like the steady state time and the validity of the Lucas–Washburn equation are examined. The results are also discussed in dimensionless form.

Keywords: Capillary rise; Capillary tube; Analytic solution; Liquid penetration; Porous medium; Imbibition; Lucas–Washburn equation; Washburn equation; Lambert W function

1. Introduction

When regarding the behavior of a liquid brought into contact with a vertical, small tube at first a fairly fast flow into it will develop. Later the rising of the liquid will continuously slow down until finally a steady state is reached. The description of the liquid rise over time by mathematical methods and its prediction are of great interest as can be seen from the following brief literature review. In 1918 Lucas [1] and 1921 Washburn [2] are the first to give an analytic explanation of the rate of liquid rise in a capillary tube. They consider a flow regime where the influence of inertia and the influence of gravity can be neglected. Marmur and Cohen [3] characterize porous media by analyzing the kinetics of capillary penetration. Ichikawa and Satoda [4] describe the interface dynamics of capillary flow and derive dimensionless variables. In 1997 Quere [5] investigates the capillary rise dominated by inertial forces and finds oscillations to occur if the fluid viscosity is low enough. In 2000 Zhmud et al. [6] give a good overview over the solutions for the different time regimes and derive short- and long-time asymptotic solutions. Siebold et al. [7] carry out capillary rise experiments in glass capillaries and packed powder to investigate the effect of the dynamic contact angle. Hamraoui and Nylander [8] provide an analytical approach for setups with a highly dynamic contact angle. In 2004 Chan et al. [9] give factors affecting the significance of gravity on infiltration of a liquid into a porous medium. Lockington and Parlange [10] find an equation for the capillary rise in porous media. Xue et al. [11] write about dynamic capillary rise with hydrostatic effects. In a recent paper Chebbi [12] investigates the dynamics of liquid penetration and compares numerical results with asymptotic solutions.

To look at the problem in more detail the momentum balance of a liquid inside a tube shall be presented. The following assumptions hold: (i) the flow is one-dimensional, (ii) no friction or inertia effects by displaced air occur, (iii) no inertia or entry effects in the liquid reservoir, (iv) the viscous pressure loss inside the tube is given by the Hagen–Poiseuille respectively the Darcy law both valid for laminar flow, and (v) the constant capillary pressure can be calculated with the static contact angle θ (see online supplementary material, Appendix 1) and the tube (or pore) radius R. With these assumptions the momentum balance of a liquid inside a capillary tube gives

$$\frac{2\sigma\cos\theta}{R} = \rho g h \sin\psi + \frac{8\mu h}{R^2}\dot{h} + \rho\frac{\mathrm{d}(h\dot{h})}{\mathrm{d}t}. \qquad (1)$$

* Corresponding author.
E-mail address: dreyer@zarm.uni-bremen.de (M. Dreyer).

0021-9797/$ – see front matter
doi:10.1016/j.jcis.2008.01.009

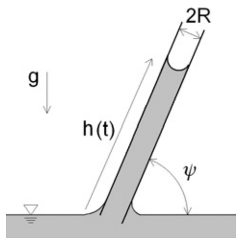

Fig. 1. Setup with an inclined tube.

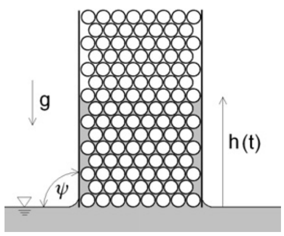

Fig. 2. Setup using a porous medium.

Here σ refers to the surface tension, R to the inner tube radius, ρ to the fluid density, g to gravity and μ to the dynamic viscosity. In Eq. (1) the individual terms refer to (left to right):

- The capillary pressure;
- The gravity term (hydrostatic pressure);
- The viscous pressure loss (Hagen–Poiseuille);
- The inertia term.

ψ (see Fig. 1) is the angle formed between the inclined tube and the free liquid surface. It shall be mentioned that for an inclined setup the height h is not the absolute height in relation to the liquid reservoir level but the distance covered within the tube.

When it comes to the momentum equation of a liquid in a generic porous medium (see Fig. 2), the Darcy law can be used. It gives the viscous pressure loss as

$$\nabla p = -\frac{\mu}{K} v_{\mathrm{s}}, \tag{2}$$

where v_{s} is the volume averaged velocity (superficial velocity) and K the permeability of the porous medium. Comparing the Hagen–Poiseuille law and Eq. (2) shows that both laws are interchangeable with each other giving

$$R^2 = \frac{8K}{\phi}, \tag{3}$$

with ϕ being the porosity of the material. The porosity is included as both laws are defined for the interstitial (Hagen–Poiseuille) and the superficial velocity (Darcy), respectively. Thus the momentum equation in a porous medium using the Darcy law reads

$$\frac{2\sigma\cos\theta}{R} = \rho g h \sin\psi + \frac{\phi\mu h}{K}\dot{h} + \rho\frac{\mathrm{d}(h\dot{h})}{\mathrm{d}t}. \tag{4}$$

2. Analytic solutions

Lucas [1] in 1918 and Washburn [2] in 1921 consider a flow regime where the influence of inertia and the influence of gravity can be neglected. They find

$$h^2 = \frac{\sigma R\cos\theta}{2\mu} t. \tag{5}$$

In the following sections the gravity term shall not be neglected which still allows to give an analytic solution as already shown by Washburn in 1921, however in terms of $t(h)$ and not $h(t)$ as we seek it. By neglecting the inertia term of Eqs. (1) or (4) (here shown for Eq. (4)) and using the constants (capillary tube and Darcy version)

$$a = \frac{\sigma R\cos\theta}{4\mu} \mathrel{\widehat{=}} \frac{2\sigma\cos\theta}{\phi\mu}\frac{K}{R} \tag{6}$$

and

$$b = \frac{\rho g R^2 \sin\psi}{8\mu} \mathrel{\widehat{=}} \frac{\rho K g \sin\psi}{\phi\mu}, \tag{7}$$

one obtains

$$\dot{h} = \frac{a}{h} - b. \tag{8}$$

As mentioned above, an analytic solution to this equation is given by Washburn [2] or Lukas and Soukupova [13]. For the initial condition $h(0) = 0$ they find the implicit analytic form

$$t = -\frac{h}{b} - \frac{a}{b^2}\ln\left(1 - \frac{bh}{a}\right), \tag{9}$$

in terms of $t = t(h)$. Hamraoui and Nylander [8] find this solution to diverge as the liquid approaches the equilibrium height. In 2000 Zhmud et al. [6] evolve a long-term asymptotic solution in terms of $h(t)$, shown here rearranged as

$$h(t) = \frac{a}{b}\left(1 - e^{-\frac{b^2 t}{a}}\right). \tag{10}$$

To obtain a more accurate solution for $h(t)$ we follow a new approach. Equation (9) is multiplied with $-b^2/a$, 1 is subtracted on both sides, and taking the power of e gives

$$-e^{-1-\frac{b^2 t}{a}} = \left(\frac{hb}{a} - 1\right) e^{\frac{hb}{a} - 1}. \tag{11}$$

At this point the Lambert W function $W(x)$ named after Johann Heinrich Lambert, and defined by an inverse exponential function

$$x = W(x)e^{W(x)} \tag{12}$$

can be used to solve for h. It can be seen that Eq. (11) follows the form

$$y(t) = x(h)e^{x(h)}. \tag{13}$$

By definition the W function can be written as

$$y(t) = W\big(y(t)\big)e^{W(y(t))}. \tag{14}$$

Relating Eq. (13) and Eq. (14) gives

$$x(h)e^{x(h)} = W\big(y(t)\big)e^{W(y(t))}. \tag{15}$$

From this it can be seen that

$$x(h) = W\big(y(t)\big). \tag{16}$$