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Love's rectangular contact problem revisited: A complete solution

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A B S T R A C T

Love's rectangular contact solution was recognized as the key ingredient in developing fast Fourier Transform related algorithms for computational contact analyses. This paper proposes an effective notation to simplify the analytical derivations, which are only carried out on the primitive functions. The complete solution of the stresses and displacements, together with the surface deflection, produced by the both uniform normal and tangential loadings over a rectangular patch are solved in a more compact and consistent way, with explicit closed-form solutions optimized for computational efficiency and numerical stability. The correlation to the Green's functions due to Boussinesq and Cerruti is also noted. The present work complements the existing literature and provides a complete reference to the classical contact solution.

1. Introduction

Contact analyses, which can be traced back to the work of Hertz [1], have a broad spectrum of applications in engineering fields, including wheel rail contact, as well as mechanisms of friction, wear and fatigue in bearings and gears. A basic problem in contact mechanics is to find the stresses and deformations produced in an elastic half-space under the action of normal and tangential tractions over a bounded area of the surface. Such a half-space contact model is important to predict the counter-conformal contact which occurs in a region sufficiently small in comparison with the realistic geometry of the contacting bodies.

The classical approach to solve the elastic field of a half-space loaded by surface tractions is based on Boussinesq-Cerruti's potential functions [2]. In 1929, Love [3] obtained a closed-form solution to the surface deflection of a half-space which is subjected to a uniform surface normal pressure over a rectangular patch. He also computed the derivatives necessary for evaluating the subsurface stress field [3]. A complementary class of problems considering surface tangential loading are also of interest to address the effects of friction. As an analogous effort to Love's solution, Ahmadi et al. [4] solved the interior stress field produced by a uniform tangential loading over a surface patch.

Although the literature search is not exhaustive, the contact mechanics community has seen a variety of publications related to the Love solution (Table 1). Except for the well-known Love's surface deflection solution, it seems that there is usually no standard notation or a universally accepted format in representing the formulation. The complete rectangular contact solution is scattered in various technical papers of contact mechanics, but has not been systematically documented even in the monographs [2,8]. Some works have only listed the necessary derivatives of the potential, and are difficult to be used as readily documented references.

Allowing for the both loading types, a complete solution set, including subsurface displacements and stresses as well as surface deflection, is desirable for a typical contact analysis. Moreover, recognition of the special structure of Love's solution should expedite the numerical computations.

The paper is organized as follows: In Section 2, the general potential theory of half-space contact is reviewed, with main results summarized for developing the analytical solutions. The numerical techniques, taking advantage of the fast Fourier transform (FFT), are also discussed, where the elementary solutions of Love's problem as the key ingredients are highlighted. A notation that captures the special feature of Love's solution is then introduced. In Section 3, we present a complete set of the closed-form solutions to Love's problem. Detailed formulae are derived for the both displacement and stress components, produced by the combination of uniform normal and tangential loadings over a rectangular patch. In Section 4, some mathematical issues are

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Nomenclature

a	Radius of the circular Hertzian contact in the benchmark example
$C_{33}^{i-k,j-l}$	Influence coefficient of the deflection due to normal pressure
diff	Any order of partial derivative with respect to x1 or x2
$F(\xi_1, \xi_2, \xi_3)$	Response primitive function
$F(\cdots)\|[\cdots]$	The notation proposed for the elementary solutions
$g_{ij}(\mathbf{x}-\mathbf{x}')$	Green's function of the stress field
$G(\xi_1, \xi_2, \xi_3)$	Green's function
$G_{ij}(\mathbf{x}-\mathbf{x}')$	Green's function of the displacement field
I_g	The set of all the grid nodes
$[i, j]$	Nodal numbering for a two dimensional discretization
$I^{(i)}$	Key primitive functions related to the elementary solution
$K^{(i)}$	Key potential functions of the elementary solution
N_1, N_2	Total number of grids in the x_1 and x_2 directions, respectively
$O(\cdot)$	Order of operation
p_H	The peak pressure of the circular Hertzian contact
$p_i(x'_1, x'_2)$	Distributed tractions along the xi directions on the surface
p_{j0}	The magnitude of the uniform loading patch
$P_j(x'_1, x'_2)$	Concentrated force applied at a surface point
Q	The common structure of elementary potentials
$Q_i^{(j)}$	Potential functions for a general distributed loading condition
r	Distance between a field point and the excitation source point
S	Loaded area
S_1	Rectangular patch with uniform traction loadings
u_H	Deflection at the origin of the circular Hertzian contact
u_i	Displacement component
(x_1, x_2, x_3)	Coordinate system
$(x'_{10}, x'_{20}, 0)$	Center of the rectangular loaded patch
$\mathbf{x}(x_1, x_2, x_3)$	A general field point of the elastic half-space
$\mathbf{x}'(x'_1, x'_2, 0)$	A general surface point within the loaded area
X_i	Auxiliary functions defined by Eq. (26).
$x_{1\alpha}$, $x_{2\beta}$	Variables for bounds of integration, cf. Eq. (14).
Δ_1, Δ_2	The side lengths of rectangular loaded patch
λ_{ij}	Primitive function of the displacement field
Λ_{ij}	Elementary influence coefficient of the stress field
μ	Shear modulus
ν	Poisson's ratio
ξ_i	Variables in Green's functions and response primitive functions
π	The ratio of the circumference of a circle to the diameter
ρ	Distance between a surface field point and the excitation point
σ_{ij}	Stress tensor
$\boldsymbol{\sigma}$, σ_i	Stress component in the form of Voigt notation
φ_{ij}	Primitive function of the displacement field
Φ_{ij}	Elementary influence coefficient of the displacement field
$\psi^{(i)}$	Auxiliary functions defined by Eq. (6)
Ω, χ	Integral kernels of the potential functions, cf. Eq. (3)

Subscripts

i	The coordinate component, or index of the Voigt notation
j	The coordinate component of the potentials and loadings
α, β	The indices with value ranged from 1 to 2.

Superscripts

(i)	Indicating the order of derivatives of the potential functions

Symbol

$-$	Denoting variable or component on the surface of the half-space

Table 1
A survey of the literature related to Love's problem.

Author	Year	Disp.		Stresses		Surface disp.		Final solutions	Notes & Comments
		N	*T*	*N*	*T*	*N*	*T*		
Love [3]	1929	√		√		√		×	Presented related derivatives, but no final solution
Kalker [5]	1979					√	√	×	Listed some primitives, but no further information
Ahmadi et al. [6]	1983					√	√	√	Showed numerical discretization and formulation
Johnson [2]	1985					√		√	Presented theoretical basis, not much final results
Kalker [7]	1986	√	√	√	√			×	Presented displacement gradient
Ahmadi et al. [4]	1987				√			√	Some stress expressions are lengthy
Hills et al. [8]	1993	√	√	√	√	√	√	Partial	No final expressions for most components
Dydo [9]	1993	√	√	√	√			×	Listed required derivatives, but no final solution
Bjorklund & Andersson [10]	1994					√	√	√	No subsurface solution
Dydo & Busby [11]	1995	√	√	√	√			×	Only presented 3 key primitives, no final solution
Liu & Wang [12]	2002			√	√			√	No detailed derivations
Chen & Wang [13]	2008					√	√	√	No detailed derivations
Willner [14]	2008					√	√	√	No subsurface solution

where symbol "*N*" stands for normal traction and "*T*" tangential traction, uniformly applied over the rectangular patch.

examined and the elastic solutions at a surface point are deduced. The correlation between the elementary solution and the related Green's functions are also discussed to validate the present work. Finally, concluding remarks are given in Section 5; and for the reader's convenience, the required derivatives of three key primitive functions for the present work are summarized in the Appendix.