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Tribology International 41 (2008) 524–534
TRIBOLOGY INTERNATIONAL
www.elsevier.com/locate/triboint

Generalized capstan problem: Bending rigidity, nonlinear friction, and extensibility effect

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Received 18 September 2006; received in revised form 1 November 2007; accepted 3 November 2007

Abstract

A theoretical model of the *capstan* problem including the extensibility and the Poisson's ratio of the rod is established in this study. Several cases were examined to investigate the effects of important parameters on the tension transmission efficiency. As a result, the rod extensibility turned out to enhance the tension ratio, competing with the effects of rod bending rigidity and the frictional behaviors of the system. In case of no frictional modification ($n = 1$, the simple Amonton's law), larger initial strain renders greater tension ratio. This effect becomes more remarkable at high radius ratio. However, the effect of the modified frictional law may oppress the effect of rod extensibility if the initial tension T_0 grows larger. The effect of Poisson's ratio also tends to increase the tension ratio. But the amount was almost negligible since the maximum decrease was at most 7.8%. We also calculated the average strain throughout the rod by solving the governing equation and iterating the value of λ_{avg}. Calculated strain throughout the rod was up to 2.6 times larger than initial strain. But this contribution leaded to at most 1.078 times larger radius ratio than the initial rod radius. Thus, it is well enough to consider only the extensibility effect on the increase of the tension ratio. We also presented several prerequisites to establish the model. Three major concerns about this model were introduced and clarified.

Keywords: Capstan equation; Extensibility; Poisson's ratio; Tension transmission; Power–law friction; Bending rigidity

1. Introduction

A tensioned rod, rope, fiber, or film wound over a circular shaped body is frequently seen in many mechanical set ups and applications. A well-known and widely used relationship is the so-called *capstan* equation [1], or Euler's equation of tension transmission. In general, the word "capstan" has the following common meanings: a rotating machine which is used to control ropes that are wound around it and used to pull/release a ship, or a rotating spindle used to move recording tape through the mechanism of a tape recorder. But the usage of the term "capstan equation" is not confined within the above two cases. In pure mechanical viewpoint, a capstan is a typical example of the physical equilibrium accompanying friction between rope or film-like solid and circular shaped body. So the capstan equation is widely used to analyze the mechanical behavior of the film/rope-like solid in contact with circular profiled surface. Rope rescue system [2,3] is a good example for applying capstan equation to their mechanical analysis. In textile area, capstan equation has been a basic equation to analyze its process. Several papers [4–8] using the capstan equation were introduced in tribological area.

Although they are significant contributions up to nowadays, all these papers are based on so called "classical capstan equation" and it should be noted that the modification of capstan equation itself is seldom introduced nor applied in the area of tribology. While it is commonly introduced and derived to help understand the mathematical procedure in all the previously refered papers, the classical capstan equation is the most simplified relationship one can derive between the incoming and outgoing tensions in the rods because it is based on the force equilibrium under such ideal condition as complete

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0301-679X/$ - see front matter
doi:10.1016/j.triboint.2007.11.005

Nomenclature

r_1	radius of the rod before extension
r_1'	radius of the rod after extension
r_2	radius of the circular shaped body
R	total radius, r_1+r_2
ρ	the radius ratio between rod and body, r_2/r_1
φ	contact angle from the start point to arbitrary point before extension
φ'	contact angle from the start point to arbitrary point after extension
ω_0	inclined angle of incoming tension T_0
ω_1	inclined angle of outgoing tension T_1
θ	total contact angle
ε	normal strain of the infinitesimal element of the rod
ν	Poisson's ratio of the rod
T	tension force
T_0	incoming/initial tension force
T_1	outgoing/result tension force
τ_A	apparent tension ratio between T_1 and T_0
τ_C	actual tension ratio, $T(\theta)/T(0)$
Q	shear force
N	normal force
F_μ	frictional force
M	bending moment
E	Young's modulus of the rod
A	cross-sectional area of the rod

close contact with no extension, flexible rod with no bending rigidity, and no frictional modification. Therefore many engineers have found that this classical *capstan* equation does not hold well for actual cases—some attempts have been made to compensate these shortcomings: Taking the bending rigidity into account [9,13–16] is the first act, whereas taking the nonlinear frictional relationship into account [11,12,16] is another one. Since the bending rigidity of the rod considered means the resistance of the rod to be bent around the drum, exerted tension is less transmitted in the presence of rod bending rigidity. In addition, the rod rigidity renders the direction of the exerted tension to be inclined inside the drum. Meanwhile, the nonlinear frictional behavior considered expresses the formula of the frictional law as power–law relationship, not linear one. Since the exponent in the formula is less than unity, it reflects the decrease of the frictional coefficient with the increase of the exerted tension.

As the most recent summary and revision of the previous attempts resolving all the previous shortcoming including Howell's contradiction (see Ref. [16]), Jung et al. [16] combined these two major corrections into one, and concluded that both factors tend to decrease the tension transmission ratio—the nonlinear frictional behavior is much more dominant than the bending rigidity. However, the initial tension should be sufficiently large so that this tension ratio decrease may actually appear.

Another important factors, but never considered before, are the extensibility and Poisson's ratio of the rod due to tension. Their effects can be very important in flexible rod such as wires or ropes. It should be noted that no published work has been made available to deal with both the extensibility and the Poisson's effects of the rod, let alone to combine them with the bending rigidity and power–law friction into a complete capstan problem solution. In most related reports, the elongation and the associated lateral contraction of the tension member were simply ignored, which in many cases, turned out to be significant negligence. Therefore, the purpose of this research is to establish a more generalized theory including such factors as extension and Poisson's ratio, in addition to the bending rigidity and power–law friction, so as to examine the extent of their influences. This work will be also useful to those who are to identify which is positive and which is not among the following factors: the extension of the rod and the decrease in the diameter due to the applied tension versus the rod bending rigidity and nonlinear frictional behavior.

2. Theoretical approach

2.1. Prerequisites before deriving governing equation

2.1.1. Direction of frictional force and contact angle measurement

Before deriving the governing equation, there exist some prerequisites we must clarify. First assigning the incoming and outgoing tensions in a rod as T_0 or T_1—Fig. 1 shows the typical situation of the bent elastic rod in contact with the capstan.

Because of the rod bending rigidity, the tension T_0 or T_1 at the both ends of the rod are no longer in the tangent direction: it splits into two components, the normal force $Q(\theta)$ bending the rod to maintain the contact with the capstan and the tangent force $T(\theta)$ pulling the rod. There exist tangential and normal components of the incoming tension as:

$$T_1^2 = T(\theta)^2 + Q(\theta)^2, \quad T(\theta) = T_1 \cos \omega_1, \quad Q(\theta) = T_1 \sin \omega_1$$

Similarly at the other end, there exist tangential and normal components of the outgoing tension as

$$T_0^2 = T(0)^2 + Q(0)^2, \quad T(0) = T_0 \cos \omega_0, \quad Q(0) = T_0 \sin \omega_0$$

An interesting feature in the above figure is that there is no definite answer which is bigger between $T(0)$ and $T(\theta)$. The key lies in the directions of the frictional force. If the frictional force goes toward $T(0)$, $T(\theta)$ is larger than $T(0)$. And consequently the contact angle is measured counter