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# Lumping evolutionary game dynamics on networks

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## HIGHLIGHTS

- Model reduction of evolutionary game dynamics with finite players organized over a network of connections.
- Impact of network topology, payoff functions and initial conditions on the asymptotic dynamics.
- Bridging lumpability for Markov chains and evolutionary games on networks.
- Dynamical interchangeability of nodes and symmetries in the model solution.

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## ABSTRACT

We study evolutionary game dynamics on networks (EGN), where players reside in the vertices of a graph, and games are played between neighboring vertices. The model is described by a system of ordinary differential equations which depends on players payoff functions, as well as on the adjacency matrix of the underlying graph. Since the number of differential equations increases with the number of vertices in the graph, the analysis of EGN becomes hard for large graphs. Building on the notion of lumpability for Markov chains, we identify conditions on the network structure allowing to reduce the original graph. In particular, we identify a partition of the vertex set of the graph and show that players in the same block of a *lumpable partition* have equivalent dynamical behaviors, whenever their payoff functions and initial conditions are equivalent. Therefore, vertices belonging to the same partition block can be merged into a single vertex, giving rise to a reduced graph and consequently to a simplified system of equations. We also introduce a tighter condition, called *strong lumpability*, which can be used to identify dynamical symmetries in EGN which are related to the interchangeability of players in the system.



## 1. Introduction

Evolutionary games have been recognized as powerful models for describing the dynamics of several natural or socio-economic phenomena (Hofbauer and Sigmund, 1998; Nowak, 2006) involving large populations of competing players. The equation commonly used in this framework is the well-known replicator equation (Weibull, 1995), which describes the dynamics under selection pressure of the share of specific phenotypes (or strategies) within a large population of identical individuals. The replicator equation is described by a system of $M$ ordinary differential equations (ODEs), where $M$ is the number of strategies available to each individual.

In many situations, populations are not homogeneous and its members can be organized in interacting groups. This is evident, for example, in the cooperation or competition of species or groups (Helbing and Johansson, 2010), or in the presence of heterogeneous agents in social systems (Borkar et al., 2004). In the context of evolutionary game theory, the multi-population replicator equation (Weibull, 1995) has been introduced to account for inhomogeneities in the population structure. In this framework, games are played across members of different groups which are characterized by different payoff functions. Similar approaches to describe the evolution of finite structured population of players are studied in van Veelen (2011), Taylor et al. (2004), Maciejewski et al. (2014), and Hauert and Imhof (2012).

The inhomogeneity of a population of individuals may also be due to the presence of specific channels of interactions among players, which are usually encoded in a network structure (Newman, 2010). It has been shown that in this case the system dynamics depends on the position of the individuals with respect to

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the whole network and on their neighborhood (Lieberman et al., 2005; Shakarian et al., 2012; Ohtsuki et al., 2007).

Recently, evolutionary games involving an underlying network (graph) of connection among players as well as different payoff functions have been studied (Ohtsuki and Nowak, 2006; Madeo and Mocenni, 2015; Szabó and Fáth, 2007; Santos et al., 2006; Jackson and Zenou, 2014). In particular, in Madeo and Mocenni (2015) the multi-population replicator equation has been extended in order to model a finite population, where players reside in the vertices of the network and play games only with their neighbors. The system of differential equations introduced in Madeo and Mocenni (2015), which we will be referring to as evolutionary game equation on networks (EGN), provides a powerful framework for modelling evolutionary behaviours with explicit heterogeneity in the number and intensity of interactions between individuals. The application of this model to a biological system involving the spatial organization of a population of bacteria under the effect of averse environmental conditions has been investigated in Madeo et al. (2014). The importance of this kind of heterogeneity relies on the possibility of modeling local interactions among finite sets of individuals. Different from the standard replicator equation, the behavior of the EGN players is not limited to pure strategies, as hypothesized in Ohtsuki and Nowak (2006), but they can behave according to mixed strategies. In the framework of the well-known prisoner's dilemma, this allows players to be, at the same time, partially cooperators and partially defectors. Moreover, the presence of the graph introduces a structure in the population, explicitly indicating who meets whom, a fact that, as will be shown in this paper, may significantly affect the system behavior.

Analogously to multi-population models, where the number of differential equations increases with the number of groups, the complexity of EGN increases with the number of vertices of the graph. This makes the mathematical analysis unfeasible for large graphs, and even numerical simulations become unwieldy. In this paper we address the following questions: Is it possible to reduce the complexity of EGN? Are there symmetries in the underlying graph which can be exploited for a more efficient analysis of EGN?

We propose a method to reduce the order of EGN which consists in identifying groups of players with similar behavior. In evolutionary games on graphs, the behaviour of each player depends on his intrinsic characteristics (such as payoff and initial conditions) as well as on his connectivity within the network. As it turns out, the network structure plays a prominent role in the evolution of the overall systems and identifying groups of players with the same intrinsic characteristics is not enough to capture symmetries at the global level. Therefore, in order to identify groups of players which behave similarly, the network structure must be taken into account. Borrowing from the notion of lumpability for Markov Chains as presented in Buchholz (1994), we employ the concept of *lumpable* and *strongly lumpable* partitions of the vertex set of a graph to identify groups of vertices with a similar structural role in the network. We then show that players belonging to the same block of a lumpable partition are dynamically equivalent if they have equivalent payoff functions and the same initial conditions, regardless of the characteristics of players belonging to different blocks. As a result vertices in the same block can be merged giving rise to a reduced graph and thus to a smaller system of differential equations. Note that our approach makes explicit use of the network structure to perform model reduction on EGN and, in this sense, integrates existing lumping techniques (Tomlin et al., 1997; Okino and Mavrovouniotis, 1998; Kuo and Wei, 1969).

Although lumpability is a purely algebraic condition, we show with some examples that our results are robust to small variations of the model parameters. Such robustness is in line with the concept of approximate lumping (Antoulas, 2005; Kuo and Wei, 1969) and with estimates on the resulting errors (Prescott and Papachristodoulou, 2012). We also show that strong lumpability can be used to identify symmetries in EGN according to the concept of symmetry in dynamical systems introduced by Golubitsky and Stewart (2006) and Stewart et al. (2003).

The method proposed in this paper is important not only because it allows to reduce the order of EGN but may also help capturing significant biological properties of the systems under investigation. Indeed, in large systems where elements interact along specific channels and behave according to selection-driven mechanisms, the evolutionary equation on graphs integrated with lumpability provide a technique to identify groups of elements with similar dynamics exploiting the connectivity among them.

*Structure of the paper*. In Section 2 we provide the details of the evolutionary game equation on networks (EGN) summarizing previous works. In Section 3 we introduce the concept of game-equivalent payoff matrices and state the equivalence of the corresponding solutions of EGN. In Section 4 we present our first main result proving that lumpable partitions allow to reduce the order of the graph and consequently to simplify EGN, and provide an example showing the robustness of our result. Our second main result is presented in Section 5, where we show that strongly lumpable partitions allows to identify dynamical symmetries in EGN. Finally, Section 6 concludes the paper. Unless otherwise stated, the proofs are relegated to the appendix.

## 2. Evolutionary game dynamics on networks

We study evolutionary games on networks, where the network structure is described by a graph $G$ over a finite set of vertices $V = \{1, \dots, N\}$. Every vertex $i \in V$ corresponds to a player and games are played only between neighboring players, i.e., player $i$ plays with player $j$ only if there is an edge connecting them in the graph. The graph can be directed or not and its edges may carry weights. We denote by $\mathbf{A}(G) = (a_{i,j})_{i,j \in V}$ the adjacency matrix of $G$. Throughout the paper, we shall assume $a_{i,j} \ge 0$ for all $i, j \in V$ (non-negative weights). The weight $a_{i,j}$ assigned to the edge from $j$ to $i$ encodes the influence of player $j$ on player $i$. Whenever $a_{i,j} > 0$ and $a_{j,i} = 0$, $j$ influences $i$ in the game, but not vice versa.

Every vertex player can choose among a finite set of *pure strategies*, namely $\mathcal{S} = \{1, \dots, M\}$. For $i \in \mathrm{V}$, $\mathbf{x}^i = \left(x_1^i, \dots, x_M^i\right)^\top$ denotes the *mixed strategy* of vertex player $i$. The value $x_s^i$ represents the probability that player $i$ uses pure strategy $s$. The set of possible mixed strategies for each player is defined as

$$\Delta_M \triangleq \left\{\mathbf{x} \in \mathbb{R}^M \middle| \sum_{s \in \mathcal{S}} x_s = 1 \text{ and } x_s \ge 0 \ \ \forall\, s \in \mathcal{S}\right\}.$$

Note that a pure strategy $s$ corresponds to the standard versor $\mathbf{e}_s \in \mathbb{R}^M$ and it is a particular case of mixed strategy. The collection of mixed strategies across all vertex players, $\mathbf{X} = \left(\mathbf{x}^1, \dots, \mathbf{x}^N\right)$, is called *mixed strategy profile* and belongs to the Cartesian product of the $M$-dimensional simplices $\Delta = \Delta_M^{|V|}$.

Every vertex player $i \in V$ has its own payoff matrix, denoted by $\mathbf{B}_i \in \mathbb{R}^{M \times M}$, which encodes the payoff earned by vertex $i$. Specifically, the entry $b_{s,r}^i$, with $i \in V$ and $s, r \in \mathcal{S}$, corresponds to the payoff that player $i$ earns when choosing pure strategy $s$ in a game where its opponent chooses pure strategy $r$. According to Madeo and Mocenni (2015), when considering evolutionary games on networks, the effective payoff earned by vertex player $i$ can be expressed as the payoff that player $i$ would earn playing a two-player game with a *virtual opponent*. In this work we slightly extend the aforementioned paper by allowing a player to play a

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