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Stability and convergence analysis of a variable order replicator–mutator process in a moving medium

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HIGHLIGHTS

- A variable order replicator–mutator process in moving medium is numerically solved.
- Stability & convergence of the applied Crank–Nicholson numerical scheme are proven.
- Transport process can be a valuable tool to control limit cycles and their amplitudes.
- Limit cycles and amplitudes depend on derivative order and stability remains unaltered.

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ABSTRACT

A more generalized approach, the concept of variable order derivative, is used to study the well-known replicator–mutator dynamics taking place in a moving medium. The biological relevance of the variable order context is explored via the language learning in social groups and stability of fixed points for the generalized model is recalled and discussed. Related graphs are plotted for different values of the derivative order γ. It happens that the threshold condition for learning accuracy symbolized by a function of payoff is a monotonically increasing function irrespective of the value of the time derivative order. Also, the limit cycles and their amplitudes are shown to vary with the value of the derivative order γ. These amplitudes become bigger as γ grows but the stability of the system is not affected. The generalized model, namely the variable order replicator–mutator dynamics in a moving medium is numerically solved via Crank–Nicholson scheme whose stability and convergence results are provided in details. An application to a variable order replicator–mutator dynamics of a population with three strategies is presented and numerical simulations are performed for some fixed values of the position variable r and the grid points. They display limit cycles appearing and disappearing in function of the values of the position r. The amplitudes of limit cycles are also proved to proportionally depend on r and the stability of the system remains unaffected. This shows the impressive effect of the transport process on the bifurcation dynamics of the model.

1. Introduction

In order to push further the ongoing investigations about the possibility of extending the well-known replicator–mutator process to the concept of variable order derivative, we provide in this paper a comprehensive analysis related to the stability and the convergence of the following variable order replicator–mutator process in a moving medium, given for a population comprising large number of agents distributed among n different types (or strategies) $S_i,\ i = 1, \ldots, n$:

$$D_t^{\gamma(r,t)} x_i = w(r)\frac{\partial x_i}{\partial r} + \sum_{j=1}^{n} x_j f_j(x) Q_{ji} - \phi(x) x_i, \quad 0 < \gamma(r,t) \le 1, \quad 0 \le r \le L, \quad t \ge 0 \tag{1.1}$$

with the initial condition

$$x_i(r, 0) = \sigma(r) \tag{1.2}$$

and boundary condition

$$x_i(0, t) = x_i(L, t) = \frac{1}{3} \tag{1.3}$$

where $T, L > 0,\ \sigma(r)$ is a real continuous function. The variable x_i is the part of the population, also seen as the frequency of each type

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$i = 1, \dots, n$ and characterized by its fitness f_i and strategy S_i. We assume that the fraction of the population $x_i = x_i(r, t)$ depends on the position r where it is located at time t. In a pure context of Biology, the spatial component r can be interpreted as follows: in population genetics for example, where x_i represents the frequency of the ith $(1 \le i \le n)$ allele of a genetic locus with n alleles, the component r can be seen as the specific spatial position (assume here to be linear) in the chromosome where that ith allele is located. Another example is the cell division process where a number of DNA or RNA biopolymer (polynucleotide) can self-replicate to yield chromosomes duplication, resulting in each separate cell to be able to have its own complete set of chromosomes. Hence, assume that there n type of such polynucleotides, and x_i represents the density of the ith $(1 \le i \le n)$ type. The component r can be interpreted as the position of that specific type in the cell or chromosome. This dynamics can take place in a moving medium (assumed to be linear) during for instance, cell division or blood flow. The velocity $w = w(r)$ of the movement is assumed to be given and depends only on the position r. The fitness, f_i, in general depends on the vector $x = (x_1, \dots, x_n)$ of the distribution of all types in the population, i.e. $f_i = g(x_1, \dots, x_n)$. The quantity Q_{ji} represents the probability that a strategy S_i is produced by another strategy S_j and then,

$$\sum_{j=1}^{n} Q_{ij} = 1, \quad 1 \le i \le n, \quad \text{for all } t, \quad r \ge 0.$$

The function

$$\phi(x(r, t)) = \sum_{j=1}^{n} x_j(r, t) f_j, \quad \text{for all } t, \quad r \ge 0$$

is the average fitness of the population and is given by the weighted average of the fitness of all the n strategies in the population in all the space $[0, L]$. It follows from this definition of $\phi(x)$ that the following conservation law holds

$$\sum_{i=1}^{n} x_i(r, t) = 1, \quad \text{for all } t, \quad r \ge 0.$$

1.1. A quick review on the concept of the variable order derivative

In the current literature, number of definitions of fractional order derivatives ranging from the local fractional derivative to variable order derivative have been proposed and investigated in numerous works (Atangana, 2015; Baleanu et al., 2012; Doungmo Goufo, 2015; Doungmo Goufo et al., 2014; Doungmo Goufo, 2014; Gorenflo and Mainardi, 2003; Khan et al., 2014; Podlubny, 1999; Miller and Ross, 1993; Hilfer, 1999; Yang et al., 2013). The variable order version has been proven to exhibit reasonable memory properties which vary along with time or spatial location (Lorenzo and Hartley, 2002; Ramirez and Coimbra, 2010; Ross and Samko, 1995; Umarov and Steinberg, 2009) and its definition results from the Riemann–Liouville integral of variable order as shown in the following definition:

Definition 1.1 (*Riemann–Liouville variable fractional order derivative*). Let $c, d > 0$ with $c < d$ and consider the function $\gamma: [c, d] \times \mathbb{R}_+ \ni (r, t) \mapsto \mathbb{R}$ such that $0 < \gamma(r, t) < 1$ and the function $h \in L_1[c, d]$. The left Riemann–Liouville integral and derivative of h of variable fractional order $\gamma(\cdot, \cdot)$ are respectively defined as

$${}_cI_t^{\gamma(\cdot,\cdot)}h(t) = \frac{d}{dt}\int_c^t \frac{1}{\Gamma(\gamma(r, t))}(t-\tau)^{\gamma(r,t)-1}h(\tau)d\tau, \quad \text{with } t > c$$

and

$${}_cD_t^{\gamma(\cdot,\cdot)}h(t) = \frac{d}{dt}\int_c^t \frac{1}{\Gamma(1-\gamma(r, t))}(t-\tau)^{-\gamma(r,t)}h(\tau)d\tau, \quad \text{with } t > c,$$

provided that ${}_cI_t^{1-\gamma(\cdot,\cdot)}h \in AC[c, d]$, where $AC[c, d]$ is the space of absolutely continuous functions on $[c; d]$. Similarly, the right Riemann–Liouville integral and derivative of h of variable fractional order $\gamma(\cdot, \cdot)$ are respectively defined as

$${}_dI_t^{\gamma(\cdot,\cdot)}{}_dI_t^{\gamma(\cdot,\cdot)}h(t) = \frac{d}{dt}\int_t^d \frac{1}{\Gamma(\gamma(r, t))}(t-\tau)^{\gamma(r,t)-1}h(\tau)d\tau, \quad \text{with } d > t$$

and

$${}_dD_t^{\gamma(\cdot,\cdot)}h(t) = \frac{d}{dt}\int_t^d \frac{1}{\Gamma(1-\gamma(r, t))}(t-\tau)^{-\gamma(r,t)}h(\tau)d\tau, \quad \text{with } d > t,$$

provided that ${}_cI_t^{1-\gamma(\cdot,\cdot)}h \in AC[c, d]$.

The previous definition yields the Caputo type derivative of variable fractional order defined as follows

Definition 1.2 (*Caputo variable fractional order derivative*). Let $c, d > 0$ with $c < d$ and consider the function $\gamma: [c, d] \times \mathbb{R}_+ \ni (r, t) \mapsto \mathbb{R}$ such that $0 < \gamma(r, t) < 1$ and the function $h \in L_1[c, d]$. The left and right Caputo derivatives of h of variable fractional order $\gamma(\cdot, \cdot)$ is defined as

$${}_cD_t^{\gamma(\cdot,\cdot)}h(t) = \int_c^t \frac{1}{\Gamma(1-\gamma(r, t))}(t-\tau)^{-\gamma(r,t)}h'(\tau)d\tau, \quad \text{with } t > c.$$

and

$${}_dD_t^{\gamma(\cdot,\cdot)}h(t) = \int_t^d \frac{1}{\Gamma(1-\gamma(r, t))}(t-\tau)^{-\gamma(r,t)}h'(\tau)d\tau, \quad \text{with } d > t,$$

provided that ${}_cI_t^{1-\gamma(\cdot,\cdot)}h \in AC[c, d]$.

Moreover, for the case $\gamma = \gamma(t)$, we have the above derivative becoming the Caputo variable order differential operator defined as:

Definition 1.3. Consider $\gamma: \mathbb{R} \ni t \mapsto \mathbb{R}_+$ be a continuous function such that $0 < \gamma(t) \le 1$, $h: \mathbb{R} \ni t \mapsto \mathbb{R}$ a continuous and differentiable function, and $H > 0$. The variable order differentiation of h in $[0, H)$ is defined as:

$$D^{\gamma(t)}h(t) = \frac{1}{\Gamma(1-\gamma(t))}\int_0^t (t-\tau)^{-\gamma(t)}h'(\tau)d\tau, \quad \text{with } t \in [0, H). \tag{1.4}$$

This operator exhibits the advantage that its derivative of a constant vanishes.

Assumed to be an extension of derivative with constant order, the concept of variable fractional order derivative exhibits unique memory properties that change according to the space location r or time t. The following simple example illustrates it easily: consider the differential equation.

$$\partial_t x(r, t) = D^{1-\gamma(t)}F(x(r, t)), \quad 0 < \gamma(t) \le 1, \quad 0 \le r \le L, \quad t \ge 0$$

Where $D^{\gamma(t)}$ is the time variable fractional order operator defined above. Then, this equation basically means that the partial derivative $\partial_t x(r, t)$ depends on part of the history of $x(r, t)$. This can easily be seen from the integral form of $D^{1-\gamma(t)}F(x(r, t))$. Then, applying the anti-derivative $D^{\gamma(t)-1}$ to both sides of the equation yields an expression more-or-less equivalent to

$$x = D^{\gamma(t)}F(x).$$

For more justification, useful properties and interpretation of this fractional derivatives in general, the readers can consult the references given at the beginning of this section. Next, the relevance of variable order derivative applied to replicator–mutator process is analyzed.