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An impulsive fishery model with environmental stochasticity. Feasibility

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ABSTRACT

An environmental random-effect over a deterministic population model of a resource (*e.g.*, a fish stock) is introduced. It is assumed that the harvest activity is concentrated at a non-predetermined sequence of instants, at which the abundance reaches a certain predetermined level, then falls abruptly by a constant capture quota (pulse harvesting). So, the abundance is modeled by a stochastic impulsive type differential equation, incorporating a standard Brownian motion in the per capita rate of growth. With this random effect, the pulse times are "stopping times" of the stochastic process. The proof of the finite expectation of the next access time, *i.e.*, the feasibility of regulation, is the main result.

1. Introduction

In the context of fishery resources, this article examines the feasibility of a management model. The amount captured is limited to a fixed quota, followed by a period of closure that lasts until a certain threshold biomass is reached, at which time fishing is again allowed with the same fixed quota, and so on. It is a mathematical continuous time model of one species, unstructured and non-deterministic. The main novelty is the combination of three rules in the dynamics of the abundance: (a) The addition to the growth rate of white noise with an amplitude proportional to the stock. (b) The consideration of an impulsive extraction at a certain threshold of the stock size. (c) A constant quota fishing policy.

Although this article focuses on the population dynamics of a fishery resource which is defined by rules (a), (b) and (c), notice that the combination of these evolutionary (or similar) rules can model many processes in a variety of contexts, such as integrated control of pests [14,15], bioreactors [10] or the treatment of diabetes [12]. So, our research question and the resolution techniques that here are shown have a wider applicability, in the framework of state-dependent feedback control models.

Concerning (a): We have hypothesized a very general unstructured stochastic version model by means of a diffusion process. It assumes that the infinitesimal increment ($\mathrm{d}N(t)$) of the abundance ($N(t)$) is the sum of underlying deterministic dynamics tendencies ($N\,r(N)dt$) plus the magnitude of the stochastic fluctuations ($\sqrt{v(N(t))}\,\mathrm{d}B(t)$). Among the sources of uncertainty to which a population may be subject, we will work with environmental stochasticity, *i.e.*, following [11], where the infinitesimal variance $v(N)$ can be modeled by σ^2N^2. The *environmental stochasticity* involves considering the randomness resulting from any change that affects the whole population and that does not diminish with its growth.

There is an abundant literature on population models in the presence of environmental noise. For models with this type of noise and capture, when the harvest is incorporated additively as a density-dependent rate which is subtracted from the natural growth, an author to consider is C.A. Braumann, see [1–4]. In [1], a quite general model for the growth of populations subjected to harvesting activities in a random environment is studied. There, conditions for non-extinction and for the existence of stationary distributions (where the noise intensity was constant or proportional to the rate of growth) are looked for. In [3] those results are generalized to density-dependent positive noise intensities of very general form.

Concerning (b): In the population models that we have cited, which combine harvesting and environmental stochasticity, the population abundance variable is of the continuous type. Here we will assume pulse harvesting between closures, which implies piecewise continuous curves for the stock, and other types of control problems.

Due to the development of techniques for locating resources and the deployability of techniques and fishing effort, the open

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harvesting intervals are considerably shorter compared to the closures. Therefore, modeling the open season as an instant is realistic enough. For impulsive harvesting see: [7–9,17–19]. Then, the amount harvested is just like a pulse in the abundance curve. A run for the resource is another explanation for an instantaneous extraction of the quota. However, although the stock captured could be previously agreed on, the run occurs when a strong competition to be first in the markets appears.

Concerning (c): The management rule assumes a continuous sampling of the abundance. The open season begins when the size of the stock reaches a certain value K^+ and continues until a catch quota Q, $0 \leq Q \leq K^+$ is achieved. Then, a new closure starts from an abundance level K_-, $K_- = K^+ - Q$, that does not compromise the survival of the resource by population uncertainty, *i.e.*, above the minimum viable population [5]. The quota is split between the different actors engaged in fishing effort and, when it is reached, a period of closure begins, lasting until the regulator again announces a level of abundance K^+. We call this regulation the *pulse constant quota fishing policy*.

Under this management, without stochasticity and with a time-autonomous growth law, models of this type have monotonic and bounded abundance curves (for values of K^+ less than the carrying capacity), which determines closures with finite length. Closed seasons with finite duration could theoretically allow bioeconomic sustainability (preservation and exploitation). However, if we consider the hypothesis of a stochastic component, coming for example from experimental factors that determine the vital rates (the biotic potential and the environmental resistance), the finiteness of the closure is not assured *a priori*. The main novel problem that guided our research questions and the results was to find conditions on the model parameters, mainly on the noise level, for the viability (finite closures) of this fishery management. Notice that in [2], in the framework of a continuous model, it is proved that in the case of a constant quota (a fixed amount harvested per unit of time) in a random environment, the population always goes to extinction.

Theorem 1 is technical in nature and aims to prove that the model is well formulated. It demonstrates the existence and uniqueness (in the probabilistic sense) of the growth curves of the resource from an initial time and level of biomass. Its Corollary 1 states that the model determines a finite population variance.

The aim of Theorem 2 is similar to that of Theorem 1, but this one proves the existence and uniqueness for all finite future time intervals.

Lastly, the main result, Theorem 3, shows that the expectation of a next opening time is finite, by assuming a condition relating the minimum per capita growth rate for population sizes under K^+ with the noise amplitude parameter.

2. Description of the model and the problem

In the first subsection we describe the regulatory basic fishery model (deterministic) and we derive the properties of the length of the closed seasons. In the second one, we will introduce stochasticity into the growth rate and formulate, with details, the research question. In both cases, the main assumptions are highlighted. In all that follows we will denote by $N(t)$ the abundance of the resource at some instant $t \in [0, \infty)$.

2.1. The deterministic model

With respect to the growth of the stock, without stochasticity, let us consider the hypothesis that follows:

H1: The *per capita rate of growth (deterministic)* is a continuous function $r : [0, \infty) \to (-\infty, +\infty)$, for which there exists a positive abundance level K, the carrying capacity, such that $r(N) > 0$ (respectively, equals 0, is less than 0) if $N < K$ (respectively, equals K, is greater than K).

Remark 1. To support **H1**, we consider the usual argument of resource limitation so that, when the population is larger, the amount of resources for individuals to survive and reproduce becomes less, and therefore death rates go up and birth rates go down, leading to negative growth rates for sufficiently large population sizes. The statistical evidence for this negative correlation between $r(\cdot)$ and $N(\cdot)$ is given in [16]. An example of $r(\cdot)$ satisfying the above condition is the *generalized logistic law*: $r(N) = r_0(1 - (N/K)^{\mu})^{\nu}$, $r_0 > 0$, $\mu, \nu \geq 1$. Hence, we have a very natural hypothesis.

If a regulator fixes a minimum level of biomass for harvesting K^+, $K^+ < K$, and a catch quota Q, the base deterministic model is

$$\begin{cases} N'(t) = r(N(t))\,N(t), & N(t) < K^+, \\ N(t^+) = K_- = K^+ - Q, & N(t) = K^+, \\ (t, N) \in [0, \infty) \times [0, K^+]. \end{cases} \tag{1}$$

where the solutions are piecewise continuous functions with continuity on the left at their discontinuities.

It is straightforward to prove, for any initial value $N(0) = K_- < K^+$, that the abundance $N(\cdot)$ is an eventually periodic trajectory that attains all the values in $[K_-, K^+]$. It is always strictly increasing except at a sequence of instants where its value is K^+, which abruptly drops by a quantity Q toward its new value K_-.

Denoting by $\{t_k\}_{k \geq 0}$ the consecutive harvest times (*i.e.*, $N(t_k) = K^+$) from the first one $t_0 \geq 0$, then $N(\cdot)$ has to satisfies the integral equation

$$N(t) = K_- \exp\left(\int_{t_k}^{t} r(N(s))ds\right), \quad t \in (t_k, t_{k+1}]. \tag{2}$$

So, substituting $t = t_{k+1}$, using $N(t_{k+1}) = K^+$ and applying the Mean Value Theorem, we have that the length of the closures in the deterministic model is given by the expression $T(K_-, K^+) = (1/r(N^*))\ln(K^+/K_-)$, for some $N^* \in (K_-, K^+)$.

In order to get some bounds, defining $a = \min_{N \in [K^-, K^+]} r(N)$ and $b = \max_{N \in [0, K^+]} r(N)$, we have

$$K_- e^{a(t-t_k)} \leq N(t) \leq K_- e^{b(t-t_k)}, \quad t \in (t_k, t-k+1], \quad k \geq 0, \tag{3}$$

and

$$\frac{1}{b}\ln\left(\frac{K^+}{K_-}\right) \leq T(K_-, K^+) \leq \frac{1}{a}\ln\left(\frac{K^+}{K_-}\right). \tag{4}$$

Note that, except for a time shift, all abundance curves are equal whatever the time-state initial condition is.

This is a non-traditional situation of sustainability (or equilibrium).

2.2. The stochastic model

We modify the deterministic model assuming the following hypothesis:

H2: The *per capita rate of growth (stochastic)* is obtained by introducing, into the first equation of the deterministic system defined by (1), a random component, more precisely we will add to the per capita rate of growth $r(\cdot)$, defined in **H1**, a noise of type $\sigma\,\mathrm{d}B(t)/\mathrm{d}t$, where $B(\cdot)$ is the standard Brownian motion, with the condition that there exists K_0, $K_- < K_0 < K^+$, such that

$$\min_{N \in [0, K_0]} \{r(N)\} > \frac{\sigma^2}{2}. \tag{5}$$

Remark 2. The inequality (5) is very natural in order to have a population dynamics not “phagocytized” by the noise. When $r(\cdot)$ is