Mathematical Biosciences 271 (2016) 128–135

Contents lists available at ScienceDirect

Mathematical Biosciences

journal homepage: www.elsevier.com/locate/mbs

Observer design for a class of nonlinear piecewise systems. Application to an epidemic model with treatment

Abdessamad Abdelhedi[a], Driss Boutat[b], Lassaad Sbita[a], Ramdane Tami[b], Da-Yan Liu[b,*]

[a] National Engineering School of Gabes, Research Unit of Photovoltaic, Wind and Geothermal Systems, Tunisia
[b] INSA Centre Val de Loire, Université d'Orléans, PRISME EA 4229, 18022 Bourges, France

ARTICLE INFO

Article history:
Received 27 January 2014
Revised 17 July 2015
Accepted 9 November 2015
Available online 19 November 2015

Keywords:
Nonlinear system
Extended observer
Epidemic model
Treatment function

ABSTRACT

Susceptible Exposed Infectious and Recovered epidemic model endowed with a treatment function (SEIR-T model) is a well-known model used to reproduce the behavior of an epidemic, where the susceptible population and the exposed population need to be estimated to predict and control the propagation of a contagious disease. This paper focuses on the nonlinear observer design for a class of nonlinear piecewise systems including SEIR-T models. For this purpose, two changes of coordinates are provided to transform the considered systems into an extended nonlinear observer normal form, on which a high gain observer can be applied. Then, the proposed method is applied to a SEIR-T model. Finally, simulation results are given to show its efficiency.

1. Introduction

The spread of a contagious disease is a complex phenomenon, since it involves thousands of individuals and uncountable contamination factors, such as wind and merchandizes. A verbal description can be used to show its global aspect. However, a mathematical model is indispensable in order to give rigorous analysis [7]. Many suitable models have been developed to reproduce the behavior of an epidemic, such as Susceptible Infectious and Recovered (SIR) model and Susceptible Exposed Infectious and Recovered (SEIR) model [23,25]. The SEIR epidemic model subdivides the population into several groups: susceptible population, exposed population, infectious population and recovered population. This model has greatly enhanced our ability to understand and study the phenomenon so as to control the spread and optimize the disease fighting actions. In 1981, a model on the interaction between fox populations and rabies was considered [2], where a quadratic control function has been integrated in the total population dynamics in order to stop the spread of rabies. Afterwards, a treatment function proportional to the number of infectious population was proposed to study bifurcation of the SIR model [19]. Then, using a similar way to [2], a quadratic treatment function was proposed [9], which takes into account not only the limits of the available resources but also the fact that they are subject to decrease. Recently, the bi-stability and the bifurcation of the SEIR model endowed with a quadratic function treatment (SEIR-T model) were studied [11].

When using a SEIR-T model, in order to predict and control the propagation of a contagious disease and the virus mutation, the future tendency of the disease needs to be reconstructed through a population and to analyze how the treatment function influences the behavior of an epidemic. For these purposes, the measurements of the susceptible population, the exposed population, the infectious population and the total population should be provided. Unlike the infectious population and the total population, the susceptible population and the exposed population are usually not measurable in a real situation. Consequently, their estimations are important. However, the existing works were basically interested in the study of the behaviors of epidemic models, such as the classification of singularities, the stability, the bifurcation, the existence of limit cycles, the chaotic behavior, etc. (see, e.g., [8,9,11,12,16,19,20]). In [1], an observer-based vaccination was designed for a SEIR epidemic model, where a control law was synthesized via an exact feedback input-output linearization approach. To the best of our knowledge, the problem of estimating the susceptible population and the exposed population in a SEIR-T model has not been studied, especially using the extended nonlinear observer normal forms concept.

The observer design is useful to estimate the state of a dynamical system. Hence, it is important in control theory. However, it is usually difficult to design an observer for a nonlinear system. In order to solve this problem, [15] proposed to transform a nonlinear system into a simple observer normal form, on which an existing observer technique can be applied. Inspired by this idea, an output observer normal form was developed in [18,21,24]. Recently, an extended

* Corresponding author. Tel.: +33 248484091.
E-mail addresses: abdessmad.abdelhedi@gmail.com (A. Abdelhedi), driss.boutat@insa-cvl.fr (D. Boutat), lassaad.sbita@enig.rnu.tn (L. Sbita), ramdane.tami@insa-cvl.fr (R. Tami), dayan.liu@insa-cvl.fr (D.-Y. Liu).

http://dx.doi.org/10.1016/j.mbs.2015.11.002
0025-5564/

nonlinear observer normal form was introduced in [14] and then developed in [3–5,17,22]. Remark that all these proposed observer normal forms can be considered as a powerful tool to design robust observers for nonlinear dynamical systems, where the states are globally estimated.

Bearing these ideas in mind, the aim of this paper is to propose an observer design to transform a SEIR-T model into an extended nonlinear observer normal form, on which a high gain observer can be applied to estimate the susceptible population and the exposed population using the infectious population and the total population. This paper is organized as follows: Section 2 provides the background of the proposed observer design by recalling the extended nonlinear observer normal form and a class of high gain observers. Section 3 introduces the considered nonlinear piecewise system, which can be transformed into an extended nonlinear observer normal form by applying two changes of coordinates. Section 4 concerns the application of the proposed method to a SEIR-T epidemic model. Then, simulation results are given to show the efficiency . Finally, some conclusions and perspectives in Section 5.

2. Background of observer design

Since nonlinear dynamical systems are usually characterized by complex structures, it is difficult to design a controller or an observer for them. In order to solve this problem, these dynamical systems are transformed into an observer normal form by applying a change of coordinates. Among the existing observer normal forms, the output depending extended nonlinear observer normal form is generally considered, which is given as follows:

$$\begin{cases} \dot{z} = A(y,w)z + \beta(y,w), \\ \dot{w} = \gamma(y,w), \\ y = Cz, \end{cases} \tag{1}$$

where $z \in \mathbb{R}^n$ and $y \in \mathbb{R}$ are respectively the real state and the real output (the measurement) of a considered dynamical system, $w \in \mathbb{R}$ is an extra output variable, $\beta(y,w) = [\beta_1(y,w), \ldots, \beta_n(y,w)]^T$, $C = [0, \ldots, 0, 1]$ is an $1 \times n$ matrix and

$$A(y,w) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \alpha_2(y,w) & 0 & \cdots & 0 & 0 \\ 0 & \alpha_3(y,w) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \alpha_n(y,w) & 0 \end{pmatrix} \tag{2}$$

is an $n \times n$ matrix.

In order to guaranty the observability of the extended nonlinear observer normal form given by (1), it is assumed that the functions $\alpha_i(\cdot,\cdot)$ for $i = 2, \ldots, n$, are nonvanishing. Then, according to [10], a high gain observer can be applied to this extended nonlinear observer normal form. Thus, it yields:

$$\begin{cases} \dot{\hat{z}} = A(y,w)\hat{z} + B(y,w) - \Gamma^{-1}(y)R_\rho^{-1}C^T(C\hat{z} - \bar{y}), \\ 0 = \rho R_\rho + G^T R_\rho + R_\rho G - C^T C, \end{cases} \tag{3}$$

where $\Gamma(y) = \mathrm{diag}[\prod_{i=2}^{n}\alpha_i(y), \prod_{i=3}^{n}\alpha_i(y), \ldots, \alpha_n(y), 1]$, and

$$G = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

The solution R_ρ of the algebraic observer equations given in (3) is explicitly given as follows:

$$R_\rho(n+1-i, n+1-j) = \frac{(-1)^{i+j}}{\rho^{i+j-1}}\binom{i+j-2}{j-1}, \tag{4}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, n$, where $\binom{i+j-2}{j-1}$ are the binomial coefficients.

It can be shown that the dynamics of the corresponding observation error is governed by the following equation:

$$\dot{e} = \dot{\hat{z}} - \dot{z} = \left(A(y,w) - \Gamma^{-1}(y)R_\rho^{-1}C^T C\right)e. \tag{5}$$

Consequently, by taking an appropriate value of ρ, the exponential stability of the observation error can be guaranteed by the boundedness of the states and the outputs (y, w).

3. A class of extended observer systems

In this paper, the following three states and single output dynamical system is considered:

$$\begin{cases} \dot{\chi}_1 = \varphi_{1,1}(y)\chi_1 + \psi_{1,2}(y)\chi_2 + \psi_{1,3}(y) + T_{1,\sigma}(y), \\ \dot{\chi}_2 = \varphi_{2,1}(y)\chi_1 + \psi_{2,2}(y)\chi_2 + \psi_{2,3}(y) + T_{2,\sigma}(y), \\ \dot{\chi}_3 = \psi_{3,1}(y)\chi_2 + \psi_{3,3}(y) + T_{3,\sigma}(y), \\ y = \chi_3, \end{cases} \tag{6}$$

where χ_i for $i = 1, 2, 3,$ are the state variables, y is the output (the measurement), σ is a switched rule, which determines the piecewise behavior of this dynamical system. It is assumed that functions $\varphi_{i,j}$ and $\psi_{i,j}$ are smooth enough. Moreover, in order to ensure the observability, it is also assumed that $\varphi_{1,1}(y) \neq 0$, $\varphi_{2,1}(y) \neq 0$ and $\psi_{3,1}(y) \neq 0$ in their domain. This means that the state variables can be written as a function of the output and its derivatives [13].

The main objective of this paper is to estimate the state variables χ_i for $i = 1, 2, 3,$ from the output (the measurement) using an observer of kind (3). In order to achieve this, the remainder of this article is devoted to address two changes of coordinates that transform the dynamical system (6) into the extended nonlinear observer normal form given by (1). Thus, an observer of kind (3) can be used in this last form. These transformations attempt to eliminate the terms containing χ_1 and χ_2 in the first dynamic equation of (6) and the term containing χ_2 in the second dynamic equation. For this purpose, we propose to successively apply two different changes of coordinates in the two following subsections.

3.1. First change of coordinates

In this subsection, we aim to annihilate the terms containing χ_2 in the two first dynamic equations given in (6). For this purpose, we have the following result.

Lemma 1. *The following change of coordinates:*

$$\xi_1 = \chi_1 - \int_0^y \frac{\psi_{1,2}(s)}{\psi_{3,1}(s)}ds, \quad \xi_2 = \chi_2 - \int_0^y \frac{\psi_{2,2}(s)}{\psi_{3,1}(s)}ds,$$

$$\text{and } \xi_3 = \int_0^y \frac{1}{\psi_{3,1}(s)}ds \tag{7}$$

transforms the dynamical system (6) into the following preliminary form:

$$\begin{cases} \dot{\xi}_1 = \varphi_{1,1}(y)\xi_1 + \varphi_{1,3}(y) + \bar{T}_{1,\sigma}(y), \\ \dot{\xi}_2 = \varphi_{2,1}(y)\xi_1 + \varphi_{2,3}(y) + \bar{T}_{2,\sigma}(y), \\ \dot{\xi}_3 = \xi_2 + \varphi_{3,3}(y) + \bar{T}_{3,\sigma}(y), \end{cases} \tag{8}$$

where

$$\varphi_{1,3}(y) = \varphi_{1,1}(y)\int_0^y \frac{\psi_{1,2}(s)}{\psi_{3,1}(s)}ds + \psi_{1,3}(y) - \frac{\psi_{1,2}(y)}{\psi_{3,1}(y)}\psi_{3,3}(y),$$

$$\bar{T}_{1,\sigma}(y) = T_{1,\sigma}(y) - \frac{\psi_{1,2}(y)}{\psi_{3,1}(y)}T_{3,\sigma}(y),$$

$$\varphi_{2,3}(y) = \varphi_{2,1}(y)\int_0^y \frac{\psi_{1,2}(s)}{\psi_{3,1}(s)}ds + \psi_{2,3}(y) - \frac{\psi_{2,2}(y)}{\psi_{3,1}(y)}\psi_{3,3}(y),$$

$$\bar{T}_{2,\sigma}(y) = T_{2,\sigma}(y) - \frac{\psi_{2,2}(y)}{\psi_{3,1}(y)}T_{3,\sigma}(y),$$