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Dynamical entropy via entropy of non-random matrices: Application to stability and complexity in modelling ecosystems

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A B S T R A C T

In the present paper we have first introduced a measure of dynamical entropy of an ecosystem on the basis of the dynamical model of the system. The dynamical entropy which depends on the eigenvalues of the community matrix of the system leads to a consistent measure of complexity of the ecosystem to characterize the dynamical behaviours such as the stability, instability and periodicity around the stationary states of the system. We have illustrated the theory with some model ecosystems.

1. Introduction

An ecosystem consisting of a large number of interacting species may be considered as a complex dynamical system [1]. In recent years dynamical system model of complex ecological system has experienced explosive growth. Different mathematical ideas and techniques have been developed in the elucidation of different underlying biological and ecological processes. To understand a complex ecosystem we need some systematic methodology. The differential equations used to model ecosystems are, in general, non-linear. It is very difficult to find out the solution of the system of non-linear equations in closed form. It is customary to study the dynamical behaviours of such systems near the stationary states. The linearized form of the system of equations around a stationary state represents the local dynamical behavior of the system. The community matrix introduced by Levin in the linearization process plays a significant role in the determination of the qualitative nature of the nearby orbits or trajectories [2]. The community matrix represents the mathematical structure of the system near the stationary states [3,4].

The objective of the present paper is to study the dynamical behaviors associated with the community matrix from a different mathematical background. The contribution of the paper is two fold: Firstly, we wish to introduce a measure of dynamical entropy for the entropic characterization of the time-evolution of the ecosystem. The methodology of the derivation of the dynamical entropy is based on the entropy of a non-random square matrix. Secondly, we wish to study the importance of the dynamical entropy in the characterization of stability, instability and periodicity of the stationary states of the ecosystem with some illustrative model ecosystems. The interrelation between the concept of stability and complexity has been investigated.

2. Model ecosystem: dynamical model and entropy

We consider a multi-species ecosystem consisting of n species with population densities $x_i(t),\ (i = 1, 2, \ldots, n)$ at any time t. Let us assume the system to be governed by the system of equations

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n, \alpha),\ (i = 1, 2, \ldots, n) \tag{2.1}$$

where α is a constant parameter. The functions f_i are assumed to be continuously differentiable in some open set $\Omega = \{x_i | x_i \geqslant 0, i = 1, 2, \ldots, n\}$. The system of model Eq. (2.1) are, in general, non-linear and difficult to find out the solution in closed form. It is customary to study such a system close to some stationary state. Let $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ be the stationary state of the system for a certain value (or a range of values) of the parameter α. We consider a small deviation about the stationary state: $y_i(t) = \delta x_i(t) = x_i(t) - x_i^*, (i = 1, 2, \ldots, n)$. Linearizing the system of Eq. (2.1) about the stationary state $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ we have the system of linear equations

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$$\dot{y}_i(t) = \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j}\right)_{x^*} y_j(t) \ (i = 1, 2, \ldots, n) = \sum_{j=1}^{n} a_{ij}(x^*) y_j(t) \tag{2.2}$$

or in matrix form

$$\dot{y}(t) = Ay(t) \tag{2.3}$$

where the Jacobian

$$A = [a_{ij}(x^*)] \tag{2.4}$$

is the so-called community matrix with elements

$$a_{ij}(x^*) = \left(\frac{\partial f_i}{\partial x_j}\right)_{x^*} \quad i,j = 1, 2, .., n \tag{2.5}$$

The elements $\{a_{ij}(x^*)\}$ of the community matrix A play significant role in the dynamical behaviours of the ecosystem [4,5].

The solution of the matrix Eq. (2.3) is given by

$$y(t) = \delta x(t) = e^{At} \delta x(0) \tag{2.6}$$

where $\delta x(0)$ is the initial deviation from the stationary state x^*. Let us now find the explicit form of the solution (2.6) in terms of the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ of the community matrix A. If the eigenvectors corresponding to the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are linearly independent then the matrix A can be converted to the form of a diagonal matrix with diagonal elements same as $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Under the mathematical conditions of linearly independence of the eigenvectors and the distinct eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, the explicit form of the solution (2.6) are given by Lakshmanan and Rajasekar [6] and Rosen [7]

$$\delta x_i(t) = \delta x_i(0) e^{\lambda_i t}, \quad (i = 1, 2, \ldots, n) \tag{2.7}$$

All the basic criteria of stability, instability and periodicity of the system follow from the solution (2.7).

Let us now try to develop an entropic theory of time-evolution of the ecosystem on the basis of the linearized model Eq. (2.3). Entropy plays a significant role in the study of evolution of a thermodynamic system [8]. Like in thermodynamics we need an expression of dynamical entropy for the study of time-evolution of the ecosystem described by the dynamical model Eq. (2.3). According to Eq. (2.3) the time-evolution of the system from the initial state to the current state is given by

$$\delta x(t) = e^{At} \delta x(0) = B(t) \delta x(0) \tag{2.8}$$

where $B(t) = e^{At}$ is the matrix of evolution $\delta x(0) \longrightarrow \delta x(t)$. The evolution matrix $B(t) = e^{At}$ characterizes the time-evolution of the system. To find out a dynamical entropy for the time-evolution $\delta x(0) \longrightarrow \delta x(t)$ we need a measure of entropy of the non-probabilistic square matrix $B(t) = e^{At}$. Following Jumarie the entropy (of order 1) of a $n \times n$ square matrix A consistant with Shannon classical entropy and Von Neumann quantum entropy is given by Jumarie [9]

$$H_1[A] = \frac{\sum_{i=1}^{n} |\lambda_i| \ln |\lambda_i|}{\sum_{i=1}^{n} |\lambda_i|} \tag{2.9}$$

This entropy $H_1[A]$ given by (2.9) provides an entropic measure of complexity (structural) of the community matrix A. The evolution matrix $B(t) = e^{At}$ is dependent on both the community matrix A and the time t. Following (2.9) the entropy (of order 1) of the evolution matrix $B(t) = e^{At}$ is given by

$$H_1[B(t)] = H_1[e^{At}] = \frac{\sum_{i=1}^{n} t \lambda_i e^{\lambda_i t}}{\sum_{i=1}^{n} e^{\lambda_i t}} \tag{2.10}$$

The entropy $H_1[B(t)]$ given by (2.10) then takes care of both the aspects of the community matrix A and the time t. We define the dynamical entropy (analogous to Kolmogorov-Sinai dynamical entropy or simply K.S. entropy) as the rate of change of the entropy $H_1[B(t)]$ or the entropy-production rate [10]

$$H(t) = \frac{H_1[B(t)]}{t} = \frac{\sum_{i=1}^{n} \lambda_i e^{\lambda_i t}}{\sum_{i=1}^{n} e^{\lambda_i t}} \tag{2.11}$$

The expression (2.11) can be written as

$$H(t) = \frac{d}{dt} \left\{ \log \sum_{i=1}^{n} e^{\lambda_i t} \right\} \tag{2.12}$$

where the sum-function $\sum_{i=1}^{n} e^{\lambda_i t}$ is the trace of the diagonal evolution matrix $B(t) = e^{At}$. This is analogous to the canonical partition-function in statistical mechanics. The dynamical entropy $H(t)$ is a real quantity inspite of the feasibility of complex or imaginary eigenvalues. Jumarie [9] has used the expression of dynamical entropy (2.12) to measure the complexity of a dynamical system. We shall consider both the terms dynamical entropy and dynamical complexity to be synonymous. In the next section we shall study the significance of the dynamical entropy (2.12) in the analysis of stability, instability and periodicity of some model ecosystems.

3. Model ecosystem: analysis of stability, periodicity and complexity

In this section we shall study the role of the measure of dynamical entropy (2.12) in the characterization of different dynamical behaviours such as stability, instability and periodicity etc. of the system around a stationary state.

Let us illustrate these with a few simple model ecosystems.

(A) Let us first consider the prey-predator model system [4,5]

$$\dot{x_1} = x_1(2 - x_1 - x_2), \quad \dot{x_2} = x_2(x_1 - x_2) \tag{3.1}$$

It has three stationary states: $(0,0)$, $(2,0)$, $(1,1)$.

(i) : Stationary point $(0,0)$; Eigenvalues $(0,2)$;

Then the dynamical complexity is given by

$$H(t) = \frac{2e^{2t}}{1 + e^{2t}} \tag{3.2}$$

which is positive and tends to 2 as $t \longrightarrow \infty$. The positive value of the dynamical complexity $H(t)$ indicates that the stationary point is non-attractive and unstable. The stationary point $(0,0)$ is thus a fixed-point repeller.

(ii) : Stationary point $(2,0)$; Eigenvalues $(2,-2)$;

The dynamical complexity now is

$$H(t) = \frac{2e^{2t} - 2e^{-2t}}{e^{2t} + e^{-2t}} \tag{3.3}$$

which is positive and tends to 2 as $t \longrightarrow \infty$. The positivity of the dynamical complexity indicates that the stationary point $(2,0)$ is non-attractive and unstable. The stationary point $(2,0)$ is then a fixed-point repeller.

(iii) : Stationary point $(1,1)$; Eigenvalues $(-1 \pm i)$;

We use the formula (2.12) to find out the dynamical complexity. The sum-function is given by

$$\sum_{i=1}^{2} e^{\lambda_i t} = e^{(-1+i)t} + e^{(-1-i)t} = e^{-t}[e^{it} + e^{-it}] = 2e^{-t} \cos t$$

Dynamical complexity is then given by

$$H(t) = \frac{d}{dt} \log \sum_{i=1}^{2} e^{\lambda_i t} = \frac{d}{dt} \log[2e^{-t} \cos t] = -[1 + \tan t] \tag{3.4}$$

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