Theoretical Population Biology 90 (2013) 29–35

Contents lists available at ScienceDirect

Theoretical Population Biology

journal homepage: www.elsevier.com/locate/tpb

Gompertz–Makeham life expectancies: Expressions and applications

Trifon I. Missov [a,b,*], Adam Lenart [a,c]

[a] Max Planck Institute for Demographic Research, Laboratory of Survival and Longevity, Konrad-Zuse-Str. 1, 18057 Rostock, Germany

[b] Faculty of Economic and Social Sciences, Institute of Sociology and Demography, University of Rostock, Ulmenstr. 69, 18051 Rostock, Germany

[c] Max Planck Odense Center on the Biodemography of Aging, Institute of Public Health, University of Southern Denmark, J. B. Winsløws Vej 9, 5000 Odense, Denmark

ARTICLE INFO

Article history:
Received 20 February 2012
Available online 29 September 2013

Keywords:
Gompertz–Makeham model
Unobserved heterogeneity
Life expectancy
Special functions
Elasticities
Exposures

ABSTRACT

In a population of individuals, whose mortality is governed by a Gompertz–Makeham hazard, we derive closed-form solutions to the life-expectancy integral, corresponding to the cases of homogeneous and gamma-heterogeneous populations, as well as in the presence/absence of the Makeham term. Derived expressions contain special functions that aid constructing high-accuracy approximations, which can be used to study the elasticity of life expectancy with respect to model parameters. Knowledge of Gompertz–Makeham life expectancies aids constructing life-table exposures.

1. Introduction

Parametric models of human mortality date back to Gompertz (1825) and his insight that death rates at adult ages increase exponentially with age. Makeham (1860) added an age-independent constant that, on the one hand, accounts for mortality that is not related to aging and moreover, statistically speaking, introduces a third parameter that improves the model fit.

In human populations, the overestimation of observed death rates at ages 80+ by the Gompertz–Makeham (GM) curve inspired the study of models that account for unobserved heterogeneity (Beard, 1959), i.e. models in which the study population is assumed to be stratified according to an unobserved measure of individual susceptibility to death. Vaupel et al. (1979) introduced a positive random variable Z, called frailty, that modulates individual hazards. The resulting marginal distribution, a continuous mixture for the baseline mortality distribution with respect to the mixing frailty distribution, describes the process at the level of the population. The simplest (in terms of frailty distribution choice) model (Vaupel et al., 1979) that accurately captures observed mortality dynamics at adult, old, and oldest-old ages (see, for example, Missov and Finkelstein (2011) and Missov and Vaupel (2013)) is the gamma-Gompertz–Makeham (ΓGM) model or its special case (when $c = 0$) the gamma-Gompertz (ΓG) model. Within its framework individual frailty Z is described by a p.d.f.

$$\pi(z) = \frac{\lambda^k}{\Gamma(k)} z^{k-1} e^{-\lambda z}, \quad k, \lambda > 0.$$

Frailty is considered to be fixed, i.e. one's frailty is initialized at the starting age of study by a value that remains the same throughout one's life. The force of mortality and the survival function of an individual with frailty $Z = z$ at age x is given, respectively, by

$$\mu(x \mid z) = z\, a e^{bx} + c \tag{1}$$

and

$$s(x \mid z) = \exp\left\{-z\,\frac{a}{b}\left(e^{bx} - 1\right) - cx\right\}, \tag{2}$$

where $a, b > 0$ are the Gompertz parameters and $c \geq 0$ stands for the level of age-independent extrinsic mortality (Kirkwood, 1985). When $c = 0$, $\mu(x \mid z)$ follows a Gompertz curve. Otherwise $\mu(x \mid z)$ has a GM shape.

The distribution of lifetimes in a ΓG mixture model is described by a survival function

$$s(x) = \int_0^\infty s(x \mid z)\, \pi(z) dz = e^{-cx}\left(1 + \frac{a}{b\lambda}(e^{bx} - 1)\right)^{-k}. \tag{3}$$

As a result remaining life expectancy at age x is expressed by the integral

$$e(x) = \int_x^\infty e^{-ct}\left(1 + \frac{a}{b\lambda}(e^{bt} - 1)\right)^{-k} dt. \tag{4}$$

* Correspondence to: Max Planck Institute for Demographic Research, Konrad-Zuse-Str. 1, 18057 Rostock, Germany.

E-mail addresses: Missov@demogr.mpg.de (T.I. Missov), alenart@health.sdu.dk (A. Lenart).

0040-5809/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.tpb.2013.09.013

In this article we focus on remaining life expectancy $e(x)$ at age x in four nested models: a Gompertz or GM baseline with (gamma-distributed) or without unobserved heterogeneity. In each of these four settings we first derive analytical expressions $e(x)$ and construct high-accuracy approximations that do not include special functions. Second, we study the elasticities of $e(x)$ with respect to all model parameters a, b, c, k, λ. Finally, we present problems, in which knowledge of Gompertz–Makeham life-expectancy expressions might be useful: (i) estimating age-specific exposures in life tables; (ii) assessing the onset of senescent mortality and the start of mortality deceleration.

2. Life expectancy: exact expressions and approximations

In this section we consider three special cases of (1) that cover the four nested models of interest: (1) when Z has a degenerate distribution concentrated at 1 and $c = 0$ (Gompertz), (2) when Z has a degenerate distribution concentrated at 1 and $c > 0$ (GM), and (3) when Z is gamma-distributed (ΓG when $c = 0$ and ΓGM when $c > 0$). In each case we first derive analytical expressions for (remaining) life expectancy and then construct high-accuracy approximations based on the properties of the resulting special functions.

2.1. Gompertz life expectancy and its approximation

In the Gompertz case, when the force of mortality is given as

$$\mu_G(x) = ae^{bx},$$

Missov and Lenart (2011) showed that the corresponding remaining life expectancy at age x can be expressed by

$$e_G(x) = \frac{1}{b}\, e^{\frac{a}{b}} E_1\left(\frac{a}{b}\, e^{bx}\right), \tag{5}$$

where $E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt$ denotes the exponential integral. As shown by Abramowitz and Stegun (1965, 5.1.11)

$$E_1(t) = -\gamma - \ln t - \sum_{n=1}^{\infty} \frac{(-1)^n\, t^n}{n \cdot n!},$$

so if $t = ae^{bx}/b$ is close to 0, then $e_G(x)$ can be approximated by

$$e_G(x) \approx \frac{1}{b}\, e^{\frac{a}{b}} \left(-\gamma - \ln\frac{a}{b}\right). \tag{6}$$

2.2. GM life expectancy and its approximation

In the Gompertz–Makeham case, when the force of mortality is given as

$$\mu_{GM}(x) = ae^{bx} + c,$$

remaining life expectancy at age x equals

$$e_{GM}(x) = \frac{1}{b}\, e^{\frac{a}{b}} \left(\frac{a}{b}\right)^{\frac{c}{b}} \Gamma\left(-\frac{c}{b}, \frac{a}{b}\, e^{bx}\right), \tag{7}$$

where $\Gamma(s, z) = \int_z^\infty t^{s-1} e^{-t} dt$ denotes the upper incomplete gamma function (see B.1). Note that

$$E_1(z) = \lim_{s\to 0} \int_z^\infty t^{s-1}\, e^{-t} dt = \lim_{s\to 0} \Gamma(s, z),$$

i.e. $e_G(x)$ is a degenerate form of $e_{GM}(x)$ when the Makeham term equals zero.

If a is close to 0, life expectancy at birth $e_{GM}(0)$ can be approximated by

$$e_{GM}(0) = \frac{1}{c} - \frac{\left(\frac{a}{b}\, e^{\gamma-1}\right)^{\frac{c}{b}}}{c\left(1 - \frac{c}{b}\right)}, \tag{8}$$

where $\gamma \approx 0.57722$ is the Euler–Mascheroni constant.

For parameter values corresponding to mortality patterns in modern societies ($0 < \frac{a}{b}\, e^{bx} \leq 1$ and $0 < \frac{c}{b} \leq 0.1$), the incomplete gamma function $\Gamma\left(-\frac{c}{b}, \frac{a}{b} e^{bx}\right)$ can be approximated by (see B.2)

$$\Gamma(s, z) = \frac{1}{s + s^2} \exp\left\{(1-\gamma)s + 0.3225 s^2\right\} - \sum_{k=0}^{\infty} (-1)^k \frac{z^{s+k}}{k!(s+k)}, \tag{9}$$

where $\zeta(n) = \sum_{k=1}^\infty k^{-n}$ is the Riemann zeta function and $0.3225 \approx \frac{\zeta(2)-1}{2}$. The closer the z-argument of the upper incomplete gamma function to 0, i.e. at younger ages, the fewer terms of $\sum_{k=0}^\infty \frac{(-1)^{k+1} z^{s+k}}{k!(s+k)}$ we need to use. To achieve a desired accuracy ε, the number of terms m in the latter series to be taken into account can be determined by

$$\frac{z^{s+m+1}}{(m+1)!(s+m+1)} \leq \varepsilon.$$

2.2.1. Example

Fitting a GM model by maximizing a Poisson likelihood (see (C.1) in Appendix C) for the 2007 United States death counts (ages 30 and above), we get the following parameter estimates: $\hat{a}_{30} = 0.00046$, $\hat{b} = 0.094$ and $\hat{c} = 0.0007$. If we want to measure remaining life expectancy at age 30, by calculating $\hat{s} = -0.0074$ and $\hat{z} = 0.0049$ from the fitted parameters and setting the error of approximation to the sum by $\varepsilon = 0.001$,

$$\frac{0.0049^{-0.0074+m+1}}{(m+1)!(-0.0074+m+1)} \leq 0.001$$

solving this inequality for m gives $m = 0.24$. Rounding up to the first integer, as only integers are allowed for k in (9), yields $m = 1$. Similarly, when the Makeham term is close to zero, in the exponential part of (9) the $\zeta(n)$-term can be left out of the approximation. In this case, the approximation error is 0.0014. Adding the two errors together, by using

$$\Gamma(s, z) = \frac{e^{(1-\gamma)s}}{s + s^2} - \frac{z^s}{s} + \frac{z^{s+1}}{s+1} \tag{10}$$

instead of (9) approximates $\Gamma(s, z)$ with an error of less than 0.0024.

To illustrate approximation quality, we calculate the exact remaining life expectancy at age 30 for the United States (2007), Japan and Germany (2009), as well as Sweden (2010) by (7), and compare it to approximation (7) taking into account (10) (see Table 1). Please note that the approximation error of (10) is inflated by the multiplicative terms preceding the upper incomplete gamma function in (7), leading to the error of not more than 0.02.

2.3. ΓGM life expectancy and its approximations

Suppose in a population the force of mortality and the survival function for an individual with frailty $Z = z$ are given by (1) and (2), respectively. Then the remaining period life expectancy at age