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Polynomial-based approximate solutions to the Boussinesq equation near a well

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ABSTRACT

This paper presents a method for constructing polynomial-based approximate solutions to the Boussinesq equation with cylindrical symmetry. This equation models water injection at a single well in an unconfined aquifer; as a sample problem we examine recharge of an initially empty aquifer. For certain injection regimes it is possible to introduce similarity variables, reducing the original problem to a boundary-value problem for an ordinary differential equation. The approximate solutions introduced here incorporate both a singular part to model the behavior near the well and a polynomial part to model the behavior in the far field. Although the nonlinearity of the problem prevents decoupling of the singular and polynomial parts, the paper presents an approach for calculating the solution based on its spatial moments. This approach yields closed-form expressions for the position of the wetting front and for the form of the phreatic surface. Comparison with a highly accurate numerical solution verifies the accuracy of the newly derived approximate solutions.

1. Introduction

This paper presents polynomial-based approximate solutions to the Boussinesq equation

$$\frac{\partial h}{\partial t} = a^2\left(\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2}\right) \tag{1}$$

in cylindrical coordinates, with a logarithmic component to model singular behavior near a wellbore. The Boussinesq equation models flows in unconfined aquifers, in which a phreatic surface exists. This nonlinear equation for the hydraulic head h rests on the Dupuit assumptions:

1. The slope and curvature of the phreatic surface are small;
2. The horizontal component of the Darcy velocity does not depend on depth.

See (Bear, 1988, section 8.1) and (Polubarinova-Kochina, 1962, section XIII) for derivations.

In cylindrical coordinates, the Boussinesq equation models aquifer flows near a wellbore, where fluid is withdrawn or recharged (Barenblatt et al., 1990). In two-dimensional flows where the wellbore radius is negligible compared to the extent of the aquifer, it is common to model the well as a point source, in which case the hydraulic head exhibits singular behavior at the well. The method presented here yields good approximations even in the presence of this behavior.

Earlier solution techniques range from the development of exact, closed-form solutions, in the tradition of Boussinesq (1904) and Polubarinova-Kochina (1962), to numerical methods (Barenblatt et al., 1990). Simple closed-form solutions, even when only approximate, can be especially useful in problems involving uncertainty or temporal fluctuations, where it may be important to use large numbers of realizations or repetitions. References Serrano and Workman (1998) and Srivastava et al. (2006) exemplify this idea using other closed-form solutions.

We consider an initially empty aquifer being recharged by a well, about which the aquifer is cylindrically symmetric. This problem was analyzed earlier in Barenblatt (1952); Li et al. (2005); Chen et al. (1995); the case of spherical symmetry was considered in Mortensen et al. (2012). One of the important features of the Boussinesq equation, which serves as a nonlinear analog of the heat equation, is that nonzero values of the hydraulic head can propagate with finite speed, in the form of a moving wetting front (Barenblatt et al., 1990; Barenblatt and Vishik, 1956). Solution methods must therefore yield accurate approximations to the position of the wetting front as well as accurate representations of the solution behind the front.

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Table 1
Physical parameters in Eq. (2).

Parameter	Physical meaning
ϕ	Porosity
g	Gravitational acceleration
k	Permeability
μ	Fluid dynamic viscosity
ρ	Density of the fluid

For homogeneous, isotropic porous media and for certain pumping or injection regimes, one can analyze Eq. (1) by introducing similarity variables (Barenblatt et al., 1990), which reduce the initial–boundary–problem for a partial differential equation to a boundary–value problem involving a nonlinear ordinary differential equation (ODE). Gilding (1992) reviews the use of similarity solutions in the context of the Boussinesq equation and its generalizations, while the comprehensive collection of the theoretical results can be found in Vazquez (2007). Barenblatt et al. (1990) solve the Boussinesq equation numerically, presenting solutions in the form of graphs for different pumping regimes. Barenblatt et al. present a numerical approach for computing the solution profile and the position of the wetting front.

The current paper constructs approximate solutions based on similarity variables and low-degree polynomials; the numerical solution is used to assess the accuracy of the approximate analytical solution. In contrast to the methods used in Li et al. (2005); Mortensen et al. (2008), where the solution is a sum of a singular term and a Taylor expansion at the wetting front, we construct an approximate solution for the ODE as the sum of a singular term and a polynomial. This polynomial contains features of solution from the whole solution domain, not just near the wetting front. We determine the coefficients of the polynomial using an approximate analytical method introduced in Lockington et al. (2000) and later extended in Telyakovskiy and Allen (2006); Telyakovskiy et al. (2002); Telyakovskiy and Kurita (2007).

The new features in this paper are (1) a method that subtracts the wellbore singularity from the nonlinear equation, an operation that modifies the equation itself, and (2) a method for finding coefficients of approximating polynomials of arbitrary degree, using conditions on higher spatial moments of the solution. In practice, we use quadratics or cubics for the approximating polynomials. For other uses of the method of moments in solving nonlinear diffusion equations in hydrology, see e.g. Lockington (1997); Prasad and Salomon (2005).

The paper has the following structure: Section 2 describes the mathematical problem; Section 3 presents the approximate solution method; Section 4 compares two polynomial-based approximate solutions with a highly accurate numerical solution; Section 5 presents conclusions.

2. Statement of the problem

Consider the flow of a single fluid in an unconfined aquifer near a well. We employ the common assumption that the flow is radially symmetric about the wellbore and satisfies the Dupuit approximation (Bear, 1988). The equation governing the hydraulic head h is then the Boussinesq Eq. (1) in cylindrical coordinates (Barenblatt, 1954):

$$\frac{\partial h}{\partial t} = a^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h^2}{\partial r}\right). \tag{2}$$

Here r denotes distance from the well; t represents time; Table 1 lists the physical parameters; and $a^2 = k\rho g/(2\phi\mu)$.

Initially the aquifer is dry:

$$h(r,0) = 0, \quad r > 0. \tag{3}$$

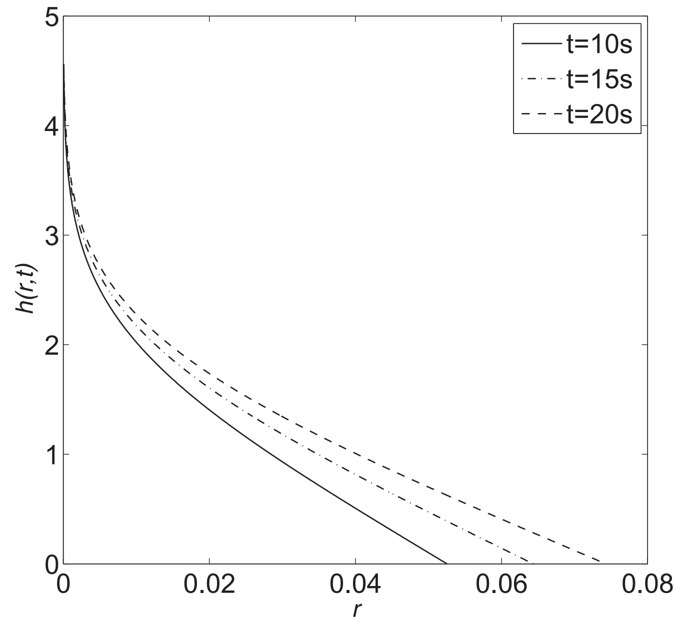

Fig. 1. Illustrative plots of $h(r, t)$ versus distance r from the well at different times.

For boundary conditions, we assume that the injection regime at the well is specified, that is, the flux at the wellbore has a prescribed, possibly time-dependent value:

$$\lim_{r\to 0} \frac{\pi k}{\mu} r \frac{\partial h^2}{\partial r} = -\tau t^{\beta}, \tag{4}$$

for fixed parameters $\tau > 0$ and $\beta \geq 0$ (Barenblatt, 1952). If recharge occurs only at the well, then

$$\lim_{r\to\infty} h(r,t) = 0, \quad t > 0. \tag{5}$$

Sample profiles of hydraulic head $h(r, t)$, showing the singular behavior, appear in Fig. 1. To obtain them we numerically solve the conditions (8)–(10), which appear below, with the method described in Appendix A, and we use the conditions (6)–(7) to obtain the profiles in the physical coordinates. For this figure we use $\beta = 0$, $\tau = 10^{-8}$ $\mathrm{m^2 s \cdot m^3/kg}$, $\phi = 0.15$, $k = 10^{-12}$ $\mathrm{m^2}$, and $\mu = 10^{-3}$ $\mathrm{kg/(m \cdot s)}$, simply for illustration.

A traditional approach to this problem is to introduce a similarity variable ξ and a scaling function $H(\xi)$ (see (Barenblatt, 1954)):

$$\xi = \frac{r}{a}\left(\frac{\pi k}{\mu\tau t^{\beta}}\right)^{1/4}\left(\frac{\beta+2}{2t}\right)^{1/2}, \tag{6}$$

$$H(\xi) = \left(\frac{\pi k}{\mu\tau t^{\beta}}\right)^{1/2} h(r,t). \tag{7}$$

Substituting (6)–(7) into (2)–(5) yields the following nonlinear ODE:

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dH^2}{d\xi}\right) + \frac{\xi}{2}\frac{dH}{d\xi} - \lambda H = 0, \quad \lambda = \frac{\beta}{\beta+2}, \tag{8}$$

subject to the boundary conditions

$$\lim_{\xi\to 0} \xi \frac{dH^2}{d\xi} = -1, \quad \lim_{\xi\to\infty} H(\xi) = 0. \tag{9}$$

The new parameter $\lambda \in [0, 1)$ parameterizes the injection regime. By solving for $H(\xi)$, one can recover $h(r, t)$ for any set of physical parameters through (6) and (7).

The initial–boundary–value problem defined by Eq. (2) together with the initial and boundary conditions (3) through (5) has three