Advances in Water Resources 90 (2016) 51–56

Contents lists available at ScienceDirect

Advances in Water Resources

journal homepage: www.elsevier.com/locate/advwatres

Multidimensional self-similar analytical solutions of two-phase flow in porous media

Radek Fučík[a,*], Tissa H. Illangasekare[b], Michal Beneš[a]

[a] Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Trojanova 13, 120 00 Prague, Czech Republic
[b] Center for Experimental Study of Subsurface Environmental Processes, Colorado School of Mines, 1500 Illinois St. Golden, CO 80401, United States

ARTICLE INFO

Article history:
Received 10 February 2016
Revised 13 February 2016
Accepted 14 February 2016
Available online 20 February 2016

Keywords:
Self-similar solution
Benchmark
Two-phase flow

ABSTRACT

In general, analytical solutions serve a useful purpose to obtain better insights and to verify numerical codes. For flow of two incompressible and immiscible phases in homogeneous porous media without gravity, one such method that neglects capillary pressure in the solution was first developed by Buckley and Leverett (1942). Subsequently, McWhorter and Sunada (1990) derived an exact solution for the one and two dimensional cases that factored in capillary effects. This solution used a similarity transform that allowed to reduce the governing equations into a single ordinary differential equation (ODE) that can be further integrated into an equivalent integral equation. We present a revision to McWhorter and Sunada solution by extending the self-similar solution into a general multidimensional space. Inspired by the derivation proposed by McWhorter and Sunada (1990), we integrate the resulting ODE in the third and higher dimensions into a new integral equation that can be subsequently solved iteratively by means of numerical integration. We developed implementations of the iterative schemes for one- and higher dimensional cases that can be accessed online on the authors' website.

© 2016 Elsevier Ltd. All rights reserved.

1. Introduction

Development of complex mathematical models of two-phase flow in porous media such as those described by Fučík and Mikyška [9,10] and Petri et al. [17] often requires versatile benchmark solutions that allow to verify numerical convergence and estimate the accuracy of the numerical method. A simplification of the domain geometry, system properties or parameters, and boundary conditions allows to derive exact (analytical or semi-analytical) solutions for the displacement of two incompressible and immiscible phases within a homogeneous or a layered heterogeneous porous medium [4,5,12]. These exact solutions not only serve as benchmark solutions, but also as effective tools to study fundamental displacement processes. A number of researchers have investigated exact solutions for cases where the gravity and/or capillarity are neglected and the exact solution of the governing equations is obtained in a form of a traveling wave such as the well-known Buckley and Leverett one dimensional analytical solution [2], generalization of the Buckley and Leverett solution by van Duijn et al. [6], or the relatively recent approach presented by Mathias et al. [14].

When the capillary effects are important, the exact solution can be found in the form of a self-similar solution as shown by Chen [3], McWhorter and Sunada [15], Sander et al. [19], Fučík et al. [11], Bjørnarå and Mathias [1] for a homogeneous porous medium and by van Duijn et al. [4], van Duijn and de Neef [5], Fučík et al. [12] for a porous medium with a single material discontinuity. For a particular functional choice of the capillary diffusion coefficient that allows to reduce the system of governing equations to the first integral, a d-dimensional exact solution can be obtained as reported previously by Sander et al. [18] and Weeks et al. [21].

This paper focuses on the self-similar solution in a homogeneous porous medium without gravity that was originally published by McWhorter and Sunada [15] and generalize its derivation to a d-dimensional space where $d \in \mathbb{N}$. This includes the self-similar solution for $d = 3$ that to the best of our knowledge has not been published in the literature and will have important practical applications in the analysis of three-dimensional numerical schemes such as convergence verification and/or estimation of the order of convergence. Even though this solution assumes the zero gravity condition that may not be realistic in three-dimensional groundwater flow, a practical application that is of relevance is in

* Corresponding author. Tel.: +420 224 358 540.
E-mail address: fucik@fjfi.cvut.cz (R. Fučík).

http://dx.doi.org/10.1016/j.advwatres.2016.02.007
0309-1708/© 2016 Elsevier Ltd. All rights reserved.

flow simulations in space. Such an application for plant irrigation in micro–gravity was discussed by Scovazzo et al. [20].

Note that in the one dimensional case, unidirectional or counter-current flows can be considered [11,12,15], but in two and higher dimensional spaces, the self-similar solution can be derived for the unidirectional displacement only [3,15]. The main idea behind the derivation of the self-similar solution is to use *a similarity substitution* to transform the governing two-phase flow equations into a single ordinary differential equation (ODE) in the d-dimensional spherical coordinates. The resulting ODE is either solved directly as proposed by Bjørnarå and Mathias [1] for $d = 1$, or transformed into an equivalent integral equation that can be solved iteratively by means of numerical integration [15] for $d = 1, 2$. We show that the derivation of the self-similar solution can be done in a general way regardless of the choice of $d \in \mathbb{N}$. For $d \geq 3$, we transform the resulting ODE into a new, general integral equation for which we propose a fast and efficient iterative solution.

The paper is organized in the following way. First, we briefly present the mathematical model and describe its transformation into the multidimensional spherical coordinates. In Section 3, we discuss the similarity transform of the governing equations into a single ODE and we highlight the important mathematical aspects of the transform that have been omitted previously in the literature but are essential in the further derivation of the self-similar solution. Then in Section 4, we present the integral approach of solving the ODE for a general dimension $d \geq 3$. In the final section, we present several typical self-similar solutions and discuss the applicability of the computational method with respect to the magnitude of the injection rate and the initial saturation.

2. Governing equations

2.1. Two-phase flow equations

We consider incompressible and immiscible flow of two phases in a homogeneous and rigid porous medium without gravity in a d-dimensional space $\mathbb{R}^d$, $d \in \mathbb{N}$. The wetting and non-wetting phases are indexed by w and n, respectively.

The continuity equation for the phase $\alpha \in \{w, n\}$ is given by

$$\phi \frac{\partial S_\alpha}{\partial t} + \nabla \cdot \vec{v}_\alpha = 0, \tag{1}$$

where ϕ [−] is the porosity, S_α [−] is the α-phase volumetric saturation, and $\vec{v}_\alpha$ $[LT^{-1}]$ is the Darcy velocity of the phase α given by

$$\vec{v}_\alpha = -\frac{k_{r,\alpha}}{\mu_\alpha} k \nabla p_\alpha, \tag{2}$$

where $k\,[L^2]$ is the intrinsic permeability and $k_{r,\alpha}$ [−], μ_α $[ML^{-1}T^{-1}]$, and p_α $[ML^{-1}T^{-2}]$ are the relative permeability, dynamic viscosity, and pressure of the phase α, respectively. By definition, $S_w + S_n = 1$.

The Eqs. (1) and (2) can be combined to express the wetting phase velocity as

$$\vec{v}_w = f(S_w)\vec{v}_T - D(S_w)\nabla S_w, \tag{3}$$

where $\vec{v}_T$ $[LT^{-1}]$ denotes the total velocity defined by $\vec{v}_T = \vec{v}_w + \vec{v}_n$, f [−] is the wetting-phase fractional flow function defined by

$$f(S_w) = \frac{\frac{k_{r,w}(S_w)}{\mu_w}}{\frac{k_{r,w}(S_w)}{\mu_w} + \frac{k_{r,n}(S_w)}{\mu_n}}, \tag{4}$$

and D $[L^2T^{-1}]$ is the capillary diffusion function given by

$$D(S_w) = -k \frac{\frac{k_{r,w}(S_w)}{\mu_w}\frac{k_{r,n}(S_w)}{\mu_n}}{\frac{k_{r,w}(S_w)}{\mu_w} + \frac{k_{r,n}(S_w)}{\mu_n}} p_c'(S_w), \tag{5}$$

where p_c $[ML^{-1}T^{-2}]$ is the capillary pressure defined by $p_c = p_n - p_w$ and p_c' denotes its first derivative with respect to S_w. In this work, we consider the following empirical models for the S_w-dependent functions

$$k_{r,w}(S_w) = S_e^{\frac{1}{2}}\left(1 - \left(1 - S_e^{\frac{1}{m}}\right)^m\right)^2,$$

$$k_{r,n}(S_w) = (1 - S_e)^{\frac{1}{3}}\left(1 - S_e^{\frac{1}{m}}\right)^{2m}, \tag{6a}$$

[16] and

$$p_c(S_w) = P_0\left(S_e^{-\frac{1}{m}} - 1\right)^{1-m} \tag{6b}$$

[13]. In Eqs. (6), m [−] and $P_0\,[Pa]$ are the fitting parameters and S_e [−] denotes the effective wetting phase saturation defined by

$$S_e = \frac{S_w - S_{w,r}}{1 - S_{w,r} - S_{n,r}}, \tag{7}$$

where $S_{\alpha,r}$ [−] denotes the residual saturation of the phase α.

Further, the continuity Eq. (1) for both phases can be transformed into

$$\nabla \cdot \vec{v}_T = 0, \tag{8a}$$

$$\phi \frac{\partial S_w}{\partial t} + \nabla \cdot (f(S_w)\vec{v}_T - D(S_w)\nabla S_w) = 0, \tag{8b}$$

where the unknown functions are the wetting phase saturation $S_w = S_w(t, \vec{x})$ and the total velocity $\vec{v}_T = \vec{v}_T(t, \vec{x})$ for all $t > 0$ and $\vec{x} \in \mathbb{R}^d$. The boundary and initial conditions will be discussed in Section 2.3.

2.2. Multidimensional spherical transform

A general multidimensional self-similar solution of the governing Eq. (8) can be obtained in the radial phase displacement flow where the wetting phase is injected through a point source placed at the origin of coordinates. Note that the complementary problem where the non-wetting phase is injected and displaces the wetting phase is described by an equation similar to Eq. (8) but with different coefficients as shown by McWhorter and Sunada [15] or Fučík et al. [11]. The derivation of the corresponding self-similar solution is analogous to the one given here.

Assuming $\vec{v}_T = \vec{v}_T(t, r)$ and $S_w = S_w(t, r)$ exhibit spherical symmetry in $\mathbb{R}^d$, where $r[L]$ denotes the non-negative radial coordinate, Eq. (8a) is resolved by

$$\vec{v}_T(t, r) = \frac{Q_0(t)}{\gamma_d r^{d-1}}\vec{\iota}, \tag{9}$$

where Q_0 $[L^dT^{-1}]$ denotes the time-dependent volumetric injection rate, $\vec{\iota}$ is the unit vector in the spherical coordinates pointing in the positive radial direction, and

$$\gamma_d = \frac{d\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2} + 1\right)}, \tag{10}$$

where Γ is the Γ-function, denotes the surface area of the d-dimensional unit sphere.

As in [15], Eq. (8b) is then transformed into

$$\gamma_d r^{d-1}\phi \frac{\partial S_w}{\partial t} + (1 - f(S_i))Q_0 \frac{\partial F}{\partial r} = 0, \tag{11}$$

where S_i[−] denotes the initial saturation. The function $F = F(t, r)$ [−] is expressed as

$$F = \frac{\frac{Q_w}{Q_0} - f(S_i)}{1 - f(S_i)}, \tag{12}$$

Download English Version:

https://daneshyari.com/en/article/6380693

Download Persian Version:

https://daneshyari.com/article/6380693

Daneshyari.com