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Rapid computation of directional wellbore drawdown in a confined aquifer via Poisson resummation

Benjamin J. Blumenthal[a,b], Hongbin Zhan[b,*]

[a] Texas Water Resources Institute, Texas A&M University, 1500 Research Parkway A110, College Station, TX 77843-2260, USA
[b] Department of Geology and Geophysics, Texas A&M University, College Station, TX 77843-2260, USA

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ABSTRACT

We have derived a rapidly computed analytical solution for drawdown caused by a partially or fully penetrating directional wellbore (vertical, horizontal, or slant) via Green's function method. The mathematical model assumes an anisotropic, homogeneous, confined, box-shaped aquifer. Any dimension of the box can have one of six possible boundary conditions: 1) both sides no-flux; 2) one side no-flux – one side constant-head; 3) both sides constant-head; 4) one side no-flux; 5) one side constant-head; 6) free boundary conditions. The solution has been optimized for rapid computation via Poisson Resummation, derivation of convergence rates, and numerical optimization of integration techniques. Upon application of the Poisson Resummation method, we were able to derive two sets of solutions with inverse convergence rates, namely an early-time rapidly convergent series (solution-A) and a late-time rapidly convergent series (solution-B). From this work we were able to link Green's function method (solution-B) back to image well theory (solution-A). We then derived an equation defining when the convergence rate between solution-A and solution-B is the same, which we termed the switch time. Utilizing the more rapidly convergent solution at the appropriate time, we obtained rapid convergence at all times. We have also shown that one may simplify each of the three infinite series for the three-dimensional solution to 11 terms and still maintain a maximum relative error of less than 10^{-14}.

1. Introduction

The analytical solution for drawdown caused by unsteady flow to a well can be derived using point sources and Green's functions (Hantush and Jacob, 1955; Gringarten and Ramey, 1973). This methodology optimized for rapid convergence at both early and late times was used to solve for heat flow by Lord Kelvin over 100 years ago (Thomson, 1884). Since then, these solutions have been continuously reorganized and republished in the heat flow literature (Carslaw and Jaeger, 1959). Although the Green's function methodology has been used in heat flow literature for quite some time, it was not until much later that it was used in well hydraulics to obtain drawdown near a pumping well (Hantush and Jacob, 1955; Gringarten and Ramey, 1973).

Early petroleum publications did not use the early-time and late-time rapidly convergent series as presented in the heat flow literature (Thomson, 1884; Carslaw and Jaeger, 1959). Many of the early solutions used in petroleum engineering employed the late-time rapidly convergent series and then attempted to find early-time approximations (Gringarten and Ramey, 1973; Clonts and Ramey, 1986; Goode and Thambynayagam, 1987; Daviau et al., 1988; Babu and Odeh, 1988; Babu and Odeh, 1989). In an effort to avoid slowly convergent series, some of these solutions restrict wellbore location. For example, Appendix A of Babu and Odeh (1988) gave drawdown approximations noting that they were only valid "for wells not too close to the boundaries"; more specifically not within 25% of the reservoir boundaries. In an effort to avoid slowly convergent series, other solutions relied on flow period approximations. For example, Goode and Thambynayagam (1987) relied on the use of four different approximations for drawdown defined by four flow periods which were "developed empirically". Given the conditions imposed and accuracies desired by these works, utilizing both early-time and late-time rapidly convergent solutions may have only required a few iterations and thus simplified their solutions (Thomson, 1884).

More recently, petroleum engineering solutions have employed both early-time and late-time rapidly convergent series, thus minimizing convergence issues (Odeh and Babu, 1990). Despite this realization in the petroleum literature, recent groundwater publications have not taken advantage of this work and are thus left with slowly convergent series at early-time (Zhan et al., 2001; Park and Zhan, 2002; Batu, 2012, 2014, 2015). Groundwater publications

* Corresponding author. Fax: +19798456162.
E-mail addresses: bnblumenthal@hotmail.com (B.J. Blumenthal), zhan@geos.tamu.edu (H. Zhan).

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Nomenclature

a, b, c	thickness of the box shaped aquifer in the x, y, and z dimensions respectively [L]
x_0, y_0, z_0	location of the point-sink inside a box shaped aquifer [L]
x, y, z	location of the point experiencing drawdown inside a box shaped aquifer [L]
K_x, K_y, K_z	principal hydraulic conductivity in the x, y, and z dimensions respectively [LT^{-1}]
S_s	specific storage [L^{-1}]
d	drawdown [L]
$Q[t]$	well discharge (positive value) or injection (negative value) rate as a function of time [L^3T^{-1}]
t	time since well discharge or injection began [T]
τ	a dummy variable representing time in an integration [T]
$\delta[]$	Dirac delta function [function]
F_x, F_y, F_z	one-dimensional boundary value solutions for the x, y, and z dimensions respectively [function]
x_1, y_1, z_1	directional wellbore starting location [L]
x_2, y_2, z_2	directional wellbore ending location [L]
x', y', z'	center point of the directional wellbore [L]
R_1, R_2, R_3	unit vector from the start to the end of a directional wellbore for the x, y, and z dimensions respectively [dimensionless]
U_1, U_2, U_3	unit vector perpendicular to both R_1, R_2, R_3 and V_1, V_2, V_3 [dimensionless]
V_1, V_2, V_3	unit vector perpendicular to both R_1, R_2, R_3 and U_1, U_2, U_3 [dimensionless]
r_w	radius of the wellbore [L]
L	length of the wellbore [L]
$x_0[\theta], y_0[\theta], z_0[\theta]$	parameterization of the point-sink along the centerline of a directional wellbore [function, L]
$x_0[w], y_0[w], z_0[w]$	parameterization of the point experiencing drawdown along the circumference of a directional wellbore at its center location [function, L]

have either approximated early-time similar to the petroleum literature of the late 1980's (Zhan et al., 2001; Park and Zhan, 2002) or have ignored the problem all together (Batu, 2012, 2014, 2015).

2. Analytical solution for drawdown caused by a directional wellbore

To address needs found in the literature review, we present drawdown solutions derived using a Green's function methodology optimized for rapid convergence at early and late times. The aquifer is assumed to be confined, homogeneous, and anisotropic. Our solutions are derived for a partially or fully penetrating, uniform flux, directional wellbore (vertical, horizontal, or slant) in a three-dimensional box shaped aquifer. Any dimension of the box-shaped aquifer can have one of six possible boundary conditions: 1) both sides no-flux; 2) one side no-flux – one side constant-head; 3) both sides constant-head; 4) one side no-flux; 5) one side constant-head; 6) free boundary conditions. The solution has been derived for calculation of drawdown within the producing wellbore, although the methodology could easily be applied to observation wells.

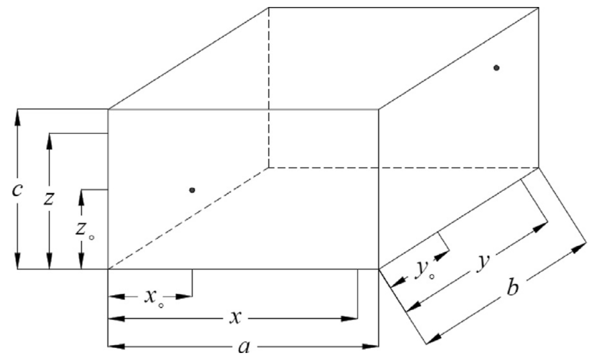

Fig. 1. Box shaped aquifer conceptual model with a point-sink (x_0, y_0, z_0) and a point experiencing drawdown (x, y, z).

Our solution addresses several key findings from the literature review. Firstly, recent groundwater solutions (Zhan et al., 2001; Park and Zhan, 2002; Batu, 2012, 2014, 2015) have not taken advantage of both the early-time and late-time rapidly convergent solutions (Thomson, 1884; Carslaw and Jaeger, 1959; Odeh and Babu, 1990). Secondly, although the work of Hantush and Jacob (1955) used source and Green's functions, their work was difficult to follow and apply, therefore an alternative approach and solution is worthwhile. Thirdly, while somewhat similar solutions to ours have been developed (Hantush and Jacob, 1955; Odeh and Babu, 1990), those solutions were constructed for the purpose of hand calculation with a much lower degree of accuracy, and have not been optimized for the speed and flexibility of numerical integration as is being considered here. Fourthly, there have been only qualitative statements made on convergence rates and the iterations required for convergence without a rigorous quantitative assessment (Thomson, 1884).

2.1. Drawdown-discharge solution for a point-sink

The mathematical relationship between a well's discharge rate and drawdown begins with the derivation of drawdown for a point-sink/source. This point-sink has a discharge rate $Q[t]$ [L^3T^{-1}] as a function of time that is positive for extraction (sink) and negative for injection (source). The origin of the Cartesian coordinate system is at left-bottom-front corner. The x and y axes are along the horizontal directions with the positive x axis pointing to the right and the positive y axis pointing to the back. The z axis is upward vertical. The point-sink may be located anywhere inside a box (Fig. 1). The dimensions [L] of the box are a, b, c for the x, y and z axes respectively. The point-sink is located at x_0, y_0, z_0 [L]. The point-sink creates aquifer drawdown at some point x, y, z [L]. The aquifer is assumed to have a uniform hydraulic head or zero drawdown everywhere before the initiation of pumping. The boundary condition is one of the six possible choices outlined previously for each of the three dimensions. The aquifer is assumed to be homogeneous but can be isotropic or anisotropic. The principal directions of the hydraulic conductivity tensor are along the x, y, and z axes respectively.

Derivation of our model begins with the partial differential equation governing confined groundwater flow with a point-sink represented by Dirac delta functions as follows

$$S_s \frac{\partial d}{\partial t} = K_x \frac{\partial^2 d}{\partial x^2} + K_y \frac{\partial^2 d}{\partial y^2} + K_z \frac{\partial^2 d}{\partial z^2} + Q[t]\delta[x - x_0]\delta[y - y_0]\delta[z - z_0], \quad (1)$$