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Stochastic-deterministic modeling of bed load transport in shallow water flow over erodible slope: Linear stability analysis and numerical simulation

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A B S T R A C T

In this article we propose a stochastic bed load transport formulation within the framework of the frictional shallow-water equations in which the sediment transport rate results from the difference between the entrainment and deposition of particles. First we show that the Saint-Venant-Exner equations are linearly unstable in most cases for a uniform base flow down an inclined erodible bed for Shields numbers in excess of the threshold for incipient sediment motion allowing us to compute noise-induced pattern formation for Froude numbers below 2. The wavelength of the bed forms are selected naturally due to the absolute character of the bed instability and the existence of a maximum growth rate at a finite wavelength when the particle diffusivity coefficient and the water eddy viscosity are present as for Turing-like instability. A numerical method is subsequently developed to analyze the performance of the model and theoretical results through three examples: the simulation of the fluctuations of the particle concentration using a stochastic Langevin equation, the deterministic simulation of anti-dunes formation over an erodible slope in full sediment-mobility conditions, and the computation of noise-induced pattern formation in hybrid stochastic-deterministic flows down a periodic flume. The full non-linear numerical simulations are in excellent agreement with the theoretical solutions. We conclude highlighting that the proposed depth-averaged formulation explains the developments of upstream migrating anti-dunes in straight flumes since the seminar experiments by Gilbert (1914).

1. Introduction

Since the earliest developments on bed load transport equations, the most common approach taken in computational river dynamics for routing sediment has involved the Exner equation and bed load transport equations, i.e. algebraic expressions relating the mean sediment flux $\bar{q}_s$ to the water conditions (e.g., the dimensionless Shields number Sh). In classical theories of sediment transport, the water flow is mostly described as a nearly uniform flow under equilibrium conditions [46,75,80]. The resulting governing equations are the one- or two-dimensional Saint-Venant equations or, more recently, the three-dimensional Navier–Stokes equations [78]. In his recent paper reviewing the last decade of research on river bed stability, Colombini [27] noted that one of the current challenges is to substitute the classic Exner equation with a probabilistic version. Stochastic partial differential equations are an emerging field not yet standard in sedimentation engineering but are becoming increasingly utilized [69,73]. In this article, we supplement the one-dimensional Saint-Venant equations with the stochastic form of the Exner equation recently proposed by Ancey and Heyman [5], we study for the first time the linear stability properties of the system and we verify the theoretical results by means of full non-linear numerical simulations of noise-induced pattern formation using a finite volume method for hybrid stochastic-deterministic flows.

A common practice in sedimentation engineering is to distinguish between flows with intense and moderate sediment transport rates [e.g. 54]. The *partial mobility regime* is encountered when $Sh < 2\,Sh_{cr}$ [42], with Sh_{cr} the critical Shields number for the onset of sediment motion. It corresponds to situations in which part of the bed sediment is mobilized by the water stream and so, the resulting sediment transport rate is low to moderate. The *full mobility regime* describes situations in which all the bed surface takes an active part in the sediment transport process, often under full bank conditions. Classical bed load transport equations such as the Meyer-Peter & Mueller, Ashida & Michiue and Fernandez Luque & van Beek formulas have

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been applied to both regimes. Applying these deterministic equations to the partial mobility regime leads to excessive errors and uncertainties in the prediction of the transported sediment volume, especially on long time scales (e.g., weeks or more) [68]. Applied to the full mobility regime, these equations usually provide the correct trend, but even in this favorable case, they cannot predict the bed load flux with an error lower than 200% [37]. We think that the improvement of bed load transport models calls for a more refined framework rather than a refinement of algebraic sediment transport equations (e.g., by increasing the number of variables of the problem). A clear impediment to greater accuracy has so far been the occurrence of wide fluctuations of the sediment transport rate and the development of bed forms. Both processes are intertwined and it is difficult to evaluate one without studying the other. In this paper, we use the stochastic framework for bed load transport proposed by Ancey et al. [4]. Within this framework, the sediment transport rate is defined as the difference between the entrainment and deposition rates and so, it can vary locally significantly over time depending on the flow conditions, bed slope, and transport of particles in the close neighborhood. This framework is well suited to the partial mobility regime as it does not directly relate the water discharge and particle flux, but defines the latter as a random variable. We will show that it is also consistent with the existing bed load transport equations for the full mobile regime: indeed, we have observed that the average sediment transport rate exhibits the same exponential dependence on the Shields number as classical formulas and other erosion-deposition models for $Sh \gg Sh_{cr}$.

Even in the simplest case of the one-dimensional Saint-Venant equations, substantial numerical difficulties arise when coupling the classical Exner equation with algebraic bed load discharge equations. For instance, two of the three system eigenvalues vanish [60], which requires a careful treatment of critical conditions and the absence of sediment motion. Schemes used for strongly coupled or decoupled numerical schemes are also the object of intense debates [29,45,66,72]. Interestingly here, we will see that the erosion-deposition formulation for bed load transport allows us to readily extend previous finite volume methods for frictional shallow water equations similar to what has been done for computing the suspended load in dilute flows, see Bohorquez and Fernández-Feria [14]. A simple strategy allows us to readily incorporate the stochastic equations for bed load transport into the numerical codes previously developed for frictional shallow-water flows on fixed beds [13,14].

The paper is organized as follows: the problem under investigation is summarized in Section 2. The linear stability properties of the mean balance equations are explored in Section 3. Next, Section 4 is devoted to the description of the hybrid finite-differences/finite-volume method and the numerical simulations. Accuracy and performance are evaluated by comparing numerical simulations with available theoretical solutions [3,4,43]. We also study the evolution of infinitesimal disturbances on a uniform base flow down an inclined plane, which leads to pattern formation and anti-dunes. Following Bohorquez [13], new analytical solutions are built and compared with numerical simulations. Conclusions are finally presented in Section 5.

2. Physical problem under investigation

2.1. Saint-Venant Exner equations: entrainment-deposition model

For one-space variable problems, the simplest morphodynamic model comprises the shallow-water (Saint-Venant) equations for the conservation of mass and momentum of the water phase and the Exner equation for the continuity equation of the bed [37]:

$$\frac{\partial h}{\partial t} + \frac{\partial h\bar{v}}{\partial x} = 0, \tag{1}$$

$$\frac{\partial h\bar{v}}{\partial t} + \frac{\partial h\bar{v}^2}{\partial x} + gh\frac{\partial h}{\partial x} = -gh\frac{\partial y_b}{\partial x} - \frac{\tau_b}{\rho} + \frac{\partial}{\partial x}\left(\nu h\frac{\partial \bar{v}}{\partial x}\right), \tag{2}$$

$$(1-\zeta_b)\frac{\partial y_b}{\partial t} = D - E \cong -\frac{\partial \bar{q}_s}{\partial x}, \tag{3}$$

in which $h(x,t) = y_s - y_b$ denotes the flow depth, $y_b(x, t)$ and $y_s(x, t)$ are the positions of the bed and free surfaces, $\bar{v}$ is the depth-averaged velocity, x is the downstream position, t is time, ϱ is the water density, τ_b is the bottom shear stress, ζ_b is the bed porosity, $\bar{q}_s$ is the average bed load transport rate (see (7) below), and D and E represent the deposition and entrainment rates, respectively. The bed slope is defined as $\tan\theta = -\partial_x y_b$. In most models based on (1)–(3), the governing equations are closed by empirical relationships for the flow resistance τ_b and sediment transport rate $\bar{q}_s$, both being functions of the flow variables $\bar{v}$ and h, and additional parameters (e.g., bed roughness and slope). Physically, this means that the sediment phase is the slave of the water phase and this dependence is justified by the momentum transfers from the water to the sediment phases [8,9]. The extra term $\partial_x(\nu h\partial_x\bar{v})$ in the momentum balance equation (2) represents a simple depth-averaged Reynolds stress [64].

2.2. Stochastic approach

Here we take a different approach to sediment transport. Following Einstein [33], we consider that sediment transport results from the imbalance between erosion and entrainment. Originally, Einstein [33] developed a Lagrangian viewpoint and expressed the erosion and deposition rates from statistical features of particle trajectories. More recently, Ancey et al. [4] used the framework of jump Markov processes for describing the random time variation in the number of moving particles n in a given volume of control. This Eulerian approach led them to express the entrainment and deposition rates as a function of the mean *particle activity* $\langle\gamma\rangle$ (i.e., the volume of moving particles n per unit bed area): $E = \lambda + \mu\langle\gamma\rangle$ and $D = \sigma\langle\gamma\rangle$. The angular brackets $\langle\gamma\rangle$ refer to the ensemble average of the random variable γ. The parameters λ, μ and σ were called the particle entrainment, the collective entrainment, and the deposition coefficients. Note the asymmetry in the expressions of the entrainment and deposition rates, which result from the differences in the physical processes involved.

As n and γ are random variables, they are characterized by their probability distribution function $P_n(x, t)$ and $P_\gamma(x, t)$. Ancey et al. [4] used the theory of birth-death Markov processes for deriving the governing equation of the number of moving particles n, more exactly its probability distribution. To make the problem more tractable, Ancey and Heyman [5] worked not with the distribution $P_n(x, t)$, but with the Poisson representation

$$P_n(x,t) = \int_{\mathbb{R}_+} P_a(x,t)\frac{e^{-a}a^n}{n!}\,\mathrm{d}a,$$

where a is called the Poisson rate and P_a is its probability distribution function. The Poisson representation can be thought of as a Laplace transform for probabilities distributions, which makes it possible to work with continuous random variables (here a) instead of discrete random variable (here n). Indeed, like in continuum mechanics, it is easier to work with local continuous variables than with global and discrete variables. Ancey and Heyman [5] introduced the particle activity as the limit of the volume occupied by the particles when the length Δx of the control volume tends to 0

$$\gamma = \lim_{\Delta x\to 0}\frac{nV_p}{B\Delta x},$$

where $V_p = \pi d^3/6$ is the typical particle volume, B is the width of the control volume, d is the mean particle diameter. Similarly, they took