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Dispersion and kinematics of multi-layer non-hydrostatic models

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ABSTRACT

Multi-layer non-hydrostatic models are gaining popularity in studies of coastal wave processes owing to the resolution of the flow kinematics, but the linear dispersion relation remains the primary criterion for assessment of model convergence. In this paper, we reformulate the linear governing equations of an N-layer model into Boussinesq form by writing the non-hydrostatic terms as high-order derivatives of the horizontal flow velocity. The equation structure allows implementation of Fourier analysis to provide a $[2N-2, 2N]$ expansion of the velocity at each layer. A variable transformation converts the governing equations into separate flux- and dispersion-dominated systems, which explicitly give an equivalent Padé expansion of the wave celerity for examination of the convergence and asymptotic properties. Flow continuity equates the depth-integrated horizontal velocity to the celerity and verifies the analytical solution. The surface-layer velocity, which is driven by the kinematic free surface boundary condition, shows a positive error and converges monotonically to the solution of Airy wave theory. When the depth parameter $kd > 2N$, flow reversal occurs in the sub-surface layers to offset overestimation of the surface velocity and to better approximate the flux. This model internal mechanism facilitates convergence of the celerity at large kd and benefits applications on wave transformation. Such non-physical flow reversal, however, might complicate studies that require detailed wave kinematics.

1. Introduction

Ocean waves comprise a broad range of wave number k that evolves with varying water depth d from the open ocean to the coast. Modeling of wave propagation from deep to shallow water requires a scalable numerical approach that can resolve both dispersion and kinematics over an applicable range of the water depth parameter kd. Recent studies have demonstrated the capability of non-hydrostatic free-surface models in describing coastal wave and circulation processes as well as their scalability with a finite number of layers to achieve desirable results for specific applications.

Multi-layer models are typically built upon the one-layer non-hydrostatic formulation, which has a relatively simple numerical framework involving up to first-order derivatives of the dependent variables (Casulli and Stelling, 1998; Stansby and Zhou, 1998). Unlike the nonlinear shallow-water equations, the non-hydrostatic formulation is capable of maintaining weak dispersion with a single layer and is ideal for studies of long waves. Zijlema and Stelling (2008) adapted the Keller box scheme of Stelling and Zijlema (2003) and the momentum-conserving scheme of Stelling and Duinmeijer (2003) to model surf- and swash-zone processes. Yamazaki et al. (2009; 2011) implemented the one-layer non-hydrostatic formulation with an upwind treatment of the flux for improved stability and subsequently developed a two-way grid nesting scheme to resolve the multi-scale processes of tsunami generation, propagation, and inundation. The resulting model, NEOWAVE (Non-hydrostatic Evolution of Ocean WAVE), is able to reproduce the 2011 Tohoku tsunami along the Japan coast as well as the dispersion across the Pacific to match current and surface elevation measurements around the Hawaiian Islands (Cheung et al., 2013; Yamazaki et al., 2013).

The dispersion and nonlinear properties improve and the applicable range expands with additional layers. Zijlema et al. (2011) developed a multi-layer model, known as SWASH (Simulating WAves till SHore), for coastal engineering applications. They demonstrated the use of two and three layers to describe wave dispersion up to $kd = 3$ and 16 respectively. Ai and Jin (2010) implemented 80 layers in their non-hydrostatic finite volume model to resolve detailed flow kinematics of a solitary wave around a submerged obstacle. They subsequently utilized 10 layers to model breaking and runup of regular and solitary waves (Ai and Jin, 2012). Ma et al. (2012) developed a multi-layer model, known as NHWAVE (Non-Hydrostatic WAVE), from the Navier Stokes equations in a terrain-following σ-coordinate system. Numerical experiments show the model with three to five layers can accurately describe wave dispersion, breaking, and runup together with longshore currents. Ma et al. (2014) then embedded

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a turbulence model in NHWAVE for studies of nearshore suspended sediment transport with 10–100 layers.

The convergence of multi-layer models can be improved with analytical or numerical treatments. Lynett and Liu (2004a; 2004b) developed a two-layer Boussinesq model and examined the convergence of the celerity and kinematics with increasing number of layers. Embedding a Boussinesq approximation in the top layer, Wu et al. (2010) found consistent improvement in dispersion with two to four layers for modeling of progressive waves over a flat bottom and a submerged bar. Bai and Cheung (2012; 2013a) proposed an integrated non-hydrostatic formulation for more effective numerical treatments of wave breaking and runup and subsequently derived a two-layer formulation with an optimized non-hydrostatic pressure distribution over the water column. The optimized model maintains the same dispersion relation as the Boussinesq formulation of Nwogu (1993) and performs numerically as a one-layer model for computational efficiency. Cui et al. (2014) proposed a variation of the optimized two-layer model with an extended applicable range of kd. Smit et al. (2013) showed the computed wave heights with 2, 3, 5, and 20 layers gradually converge to the laboratory measurements of a spilling breaker by Ting and Kirby (1994) and proposed a hydrostatic approximation at the breaking wave front to model surf-zone dynamics and circulations with a reduced number of layers.

Most of the aforementioned works focus on numerical model development and implementation. Model convergence is assessed through the linear dispersion relation either numerically or analytically with a finite number of layers. Bai and Cheung (2013b) elaborated an analytic procedure to derive the linear dispersion relation as well as the nonlinear sub- and super-harmonic wave amplitude from the integrated multi-layer formulation. With proper optimization of a piecewise linear profile, a two- and a three-layer model can give better performance in terms of the nonlinear properties comparing to the Boussinesq formulations of Wei et al. (1995) and Gobbi et al. (2000). The convergence of the flow kinematics, however, has not been systematically investigated despite the increasing popularity of multi-layer models. Accurate prediction of particle velocity is important to studies of surf-zone processes and sediment transport. As the dispersion properties improve, these models are being applied to intermediate and deep-water conditions straining the multi-layer structure with the large velocity gradient near the surface.

The multi-layer non-hydrostatic approach with the piecewise linear velocity profile has gained popularity due to its simple numerical framework and scalability in modeling complex wave dynamics and coastal circulations. However, the limited understanding of the convergence pattern related to the flow kinematics hampers its general application and obscures interpretation of the model results. The absence of proper guidelines may lead to usage of an insufficient or excessive number of layers that compromises model performance and computational efficiency. Based on the analytic procedure of Bai and Cheung (2013b) and the assumption of small amplitude periodic waves, we derive the dispersion relation and velocity profile of an N-layer non-hydrostatic model to investigate their inter-relationship. The analytical solution allows examination of convergence in comparison to Airy wave theory and the asymptotic behavior as N and kd $\to \infty$. A parametric analysis of the velocity profile and dispersion relation in terms of N and kd reveals the solution properties and provides guidance for implementation of the multi-layer approach in coastal wave modeling.

2. Linearized governing equations

The governing equations for multi-layer non-hydrostatic free-surface flow are well established in the technical literature (Ai et al., 2011; Bai and Cheung, 2012; Zijlema and Stelling, 2005). Common in their formulations is a staggered grid system and an implicit assumption of a piecewise linear velocity profile, which provides a basic approximation of the flow kinematics for illustration of the inherent convergence properties. The non-dimensional form presented by Bai and Cheung (2013b) provides a framework to systematically identify the linear component for investigation of the fundamental properties. We first convert the linear system into Boussinesq form through a direct formulation, which expresses the vertical velocity and non-hydrostatic pressure in terms of the high-order derivatives of the horizontal velocity. A linear transformation maps the velocity profile into a depth-averaged value and a series of inter-layer variations in the integrated formulation for derivation of the dispersion relation.

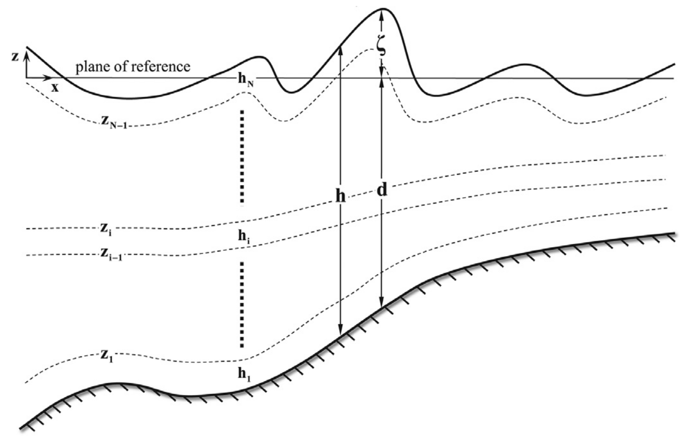

Fig. 1. Definition sketch of an N-layer free-surface flow from Bai and Cheung (2013b).

2.1. Multi-layer system in Boussinesq form

Consider a two-dimensional free-surface flow with a Cartesian coordinate system (x, z) in Fig. 1. The inviscid and incompressible flow over the varying bottom evolves with time t. The variables d and ζ denote the water depth and surface elevation and $h = \zeta + d$ is the flow depth. Decomposition of the water column into N layers provides a means to resolve the flow structure. The layer thickness h_i is variable, but its ratio to the flow depth is a constant value defined *a priori* as

$$r_i = h_i/h \quad i = 1, \ldots, N. \tag{1}$$

This simplifies the multi-layer formulation by expressing the layer interface in terms of the flow depth as

$$z_i = \sum_{j=1}^{i} r_j h - d \quad i = 0, \ldots, N. \tag{2}$$

The concept is analogous to the σ-coordinate, but z_i is independent of the vertical coordinate z. At the two limits of the index i, the interfaces z_0 and z_N correspond to the bottom and free surface as indicated in Fig. 1.

Bai and Cheung (2013b) derived the non-dimensional governing equations for the multi-layer system and utilized the parameters μ and ϵ to identify the various terms that contribute to dispersion and nonlinearity. The continuity equation and the horizontal and vertical momentum equations for layer i read

$$\frac{\partial h_i u_i}{\partial x} - u_{z_i}\frac{\partial z_i}{\partial x} + u_{z_{i-1}}\frac{\partial z_{i-1}}{\partial x} + w_{z_i} - w_{z_{i-1}} = 0 \tag{3}$$

$$\frac{\partial u_i}{\partial t} + \frac{\epsilon}{h_i}\left(\frac{\partial h_i u_i^2}{\partial x} - u_i\frac{\partial h_i}{\partial t}\right) + \frac{\partial \zeta}{\partial x} + \frac{\epsilon}{h_i}u_{z_i}\Delta w_{z_i} - \frac{\epsilon}{h_i}u_{z_{i-1}}\Delta w_{z_{i-1}} + \frac{\mu^2}{h_i}\left\{\frac{1}{2}\frac{\partial h_i(q_{z_i} + q_{z_{i-1}})}{\partial x} - \left(q_{z_i}\frac{\partial z_i}{\partial x} - q_{z_{i-1}}\frac{\partial z_{i-1}}{\partial x}\right)\right\} = 0 \tag{4}$$