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On the dynamical influence of ocean eddy potential vorticity fluxes

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ABSTRACT

The impact of eddy potential vorticity fluxes on the dynamical evolution of the flow is obscured by the presence of large and dynamically-inert rotational fluxes. However, the decomposition of eddy potential vorticity fluxes into rotational and divergent components is non-unique in a bounded domain and requires the imposition of an additional boundary condition. Here it is proposed to invoke a one-to-one correspondence between divergent eddy potential vorticity fluxes and non-divergent eddy momentum tendencies in the quasi-geostrophic residual-mean equations in order to select a unique divergent eddy potential vorticity flux. The divergent eddy potential vorticity flux satisfies a zero tangential component boundary condition. In a simply connected domain, the resulting divergent eddy potential vorticity flux satisfies a powerful optimality condition: it is the horizontally oriented divergent flux with minimum L^2 norm. Hence there is a well-defined sense in which this approach removes as much of the dynamically inactive eddy potential vorticity flux as possible, and extracts an underlying dynamically active divergent eddy potential vorticity flux. It is shown that this approach leads to a divergent eddy potential vorticity flux which has an intuitive physical interpretation, via a direct relationship to the resulting forcing of the mean circulation.

1. Introduction

The fundamental dynamical equations for the ocean can typically be cast into a flux form in which changes to physical quantities depend upon the divergence of their flux. This reflects the existence of integral conservation laws and yields a natural physical interpretation in terms of the transport of properties such as heat, salinity, or potential vorticity from one region of the ocean to another. However, in practice, the direct analysis of the dynamical impact of oceanic fluxes is often obscured by the existence of large *non-divergent* flux components, which necessarily have no direct dynamical effect. The general resolution of this issue is through the application a Helmholtz decomposition, separating the flux into a divergent component, which is dynamically active, and a non-divergent component, which is dynamically inert. Unfortunately, this decomposition is inherently non-unique in bounded domains, and is dependent upon a choice of boundary conditions (Fox-Kemper et al., 2003).

This issue is of particular concern in the analysis and comparison of eddy parameterisations, which typically specify parameterised eddy fluxes. For example, the existence of locally up-gradient fluxes does not necessarily rule out the application of down-gradient flux parameterisations – an appropriately defined divergent eddy flux may be more closely aligned counter to the mean gradients (e.g. Marshall and Shutts, 1981). Similarly, down-gradient potential vorticity closures violate momentum conservation constraints in general (Bretherton, 1966; Marshall et al., 2012), but momentum conservation can be restored via the introduction of an appropriate non-divergent eddy potential vorticity flux (Eden, 2010).

In a domain average sense, eddy potential vorticity fluxes must be oriented down the mean gradient in order to ensure net generation of eddy enstrophy, itself required in order to balance small-scale enstrophy dissipation. However it has long been recognised that this principle need not hold locally (Harrison, 1978; Holland and Rhines, 1980). Local fluxes of eddy enstrophy permit the eddy potential vorticity flux to be oriented in any direction. In Marshall and Shutts (1981) eddy fluxes are separated into a component balancing the mean advection and a residual component. In the barotropic vorticity model of Marshall (1984) it is found that the residual eddy potential vorticity flux thus defined is more strongly aligned with the mean potential vorticity gradient. The methodology is directly generalised in Nakamura (1998) and Nakamura and Chao (2002). A related approach is described in Greatbatch (2001) and Medvedev and Greatbatch (2004), whereby the eddy fluxes are separated into advective, diffusive, and rotational fluxes, which are then related to the

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components of the total (mean plus eddy) advective eddy variance flux in the along and across mean gradient directions. This results in a decomposition similar to the Temporal Residual Mean I formulation of McDougall and McIntosh (1996) (with the latter replacing the use of the total advective eddy variance flux with the mean advective eddy variance flux – see Maddison and Marshall (2013, Appendix B) for further details). The decomposition of Medvedev and Greatbatch (2004) is itself generalised in Eden et al. (2007) via the consideration of higher order eddy budgets.

An alternative approach is to decompose eddy potential vorticity fluxes into rotational and divergent components via the use of a Helmholtz decomposition (e.g. Lau and Wallace, 1979). In Roberts and Marshall (2000) ocean eddy fluxes, diagnosed from a primitive equation model, are decomposed into rotational and divergent components via the application of a Helmholtz decomposition, subject to zero normal divergent flux boundary conditions. The resulting divergent eddy fluxes are in this case found to be rather poorly correlated with corresponding mean gradients.

Non-uniquness of the Helmholtz decomposition in bounded domains, and consequences for the decomposition of eddy fluxes, is discussed at length in Fox-Kemper et al. (2003). The zero normal divergent flux boundary condition is only one of countless valid options. Subject to an alternative choice of boundary conditions it is possible, in a bounded domain, to extract an eddy flux which has a minimum norm, or a minimum deviation from the mean gradient (Fox-Kemper et al., 2003). Without any additional constraints on the problem there is no way to select a boundary condition from amongst these options.

This article discusses a physically motivated approach for resolving this ambiguity in the Helmholtz decomposition of eddy potential vorticity fluxes. Specifically the quasi-geostrophic residual-mean equations allow the identification of a one-to-one correspondence between divergent eddy potential vorticity fluxes and non-divergent eddy momentum tendencies. The definition of the latter leads to an unambiguous definition of the former, which leads to a unique divergent eddy potential vorticity flux which satisfies a zero tangential component boundary condition. In a simply connected domain the resulting divergent eddy potential vorticity flux satisfies a powerful optimality condition: it is the (horizontally oriented) divergent flux with minimum L^2 norm. Hence there is a well-defined sense in which this approach removes as much of the dynamically inactive non-divergent eddy potential vorticity flux as possible, and extracts an underlying dynamically active divergent eddy potential vorticity flux. It is shown that this approach leads to a divergent eddy potential vorticity flux which has an intuitive physical interpretation, via a direct relationship to the resulting forcing of the mean circulation.

The paper proceeds as follows. Section 2 describes the mathematical formulation. The quasi-geostrophic residual-mean equations are outlined, and the relationship between divergent potential vorticity fluxes and non-divergent momentum tendencies is described. A stream function tendency, or “force function”, is used to define the divergent potential vorticity fluxes, and it is shown that in a simply connected domain the resulting divergent potential vorticity flux satisfies an optimality property. Resulting divergent eddy potential vorticity fluxes are diagnosed from a three layer quasi-geostrophic model in Section 3. The decomposition is compared against the more conventional use of zero normal divergent potential vorticity flux boundary conditions, and the utility for the assessment of eddy parameterisations is considered. The paper concludes in Section 4.

2. Formulation

This section describes the Helmholtz decomposition of arbitrary potential vorticity fluxes into divergent and non-divergent components. Section 2.1 describes the horizontal Helmholtz decomposition, and discusses the origin of ambiguity in decomposing vector fields into divergent and rotational components. Section 2.2 introduces the quasi-geostrophic residual-mean equations, and uses these to the relate divergent potential vorticity fluxes to non-divergent momentum tendencies. In Section 2.3 this relation is used to define a horizontal Helmholtz decomposition for potential vorticity fluxes, by relating the divergent component of potential vorticity fluxes to stream function tendencies, or “force functions”, associated with momentum tendencies. The assertion that the decomposition should be linear defines a unique horizontal Helmholtz decomposition for the eddy potential vorticity flux. Finally in Section 2.4 it is shown that, in a simply connected domain, the resulting divergent eddy potential vorticity flux is optimal, in that it is the unique (horizontally aligned) divergent eddy potential vorticity flux with minimal L^2 norm. The resulting diagnostic equations for force functions are summarised in Section 2.5.

2.1. Horizontal Helmholtz decomposition

The Helmholtz decomposition of a vector field splits the field into three components: a divergent component (with zero curl), a rotational component (with zero divergence), and a harmonic component (with both zero curl and zero divergence). This article considers the horizontal Helmholtz decomposition which, for a vector field $\mathbf{F}$, takes the form:

$$\mathbf{F} = \nabla_H \Phi_F + \hat{\mathbf{z}} \times \nabla_H \Psi_F + \mathbf{H}_F, \tag{1}$$

where Φ_F and Ψ_F are two scalar potentials, the divergent component is $\nabla_H \Phi_F$, the rotational component is[1] $\hat{\mathbf{z}} \times \nabla_H \Psi_F$, and the harmonic component is $\mathbf{H}_F$. $\mathbf{H}_F$ has both zero divergence and zero horizontal curl, $\nabla_H \cdot \mathbf{H}_F = (\hat{\mathbf{z}} \times \nabla_H) \cdot \mathbf{H}_F = 0$. $\nabla_H = (\partial_x, \partial_y, 0)^T$ is the horizontal gradient operator, and $(\hat{\mathbf{z}} \times \nabla_H) \cdot (\ldots)$ is the horizontal curl operator.

A horizontal Helmholtz decomposition of $\mathbf{F}$ can in principle be performed by solving for the two potentials Φ_F and Ψ_F, and then using these to compute the harmonic residual $\mathbf{H}_F$. Taking the divergence and horizontal curl of $\mathbf{F}$ leads to two elliptic problems for the potentials:

$$\nabla_H^2 \Phi_F = \nabla_H \cdot \mathbf{F} \tag{2a}$$

$$\nabla_H^2 \Psi_F = (\hat{\mathbf{z}} \times \nabla_H) \cdot \mathbf{F}. \tag{2b}$$

The critical issue here is that no boundary conditions have been imposed on these problems. The selection of alternative boundary conditions allows harmonic fields to be exchanged between the divergent, rotational, and harmonic components of the decomposition. Without the specification of appropriate boundary conditions (e.g. as discussed in Denaro (2003)) the Helmholtz decomposition of a vector field is, in a bounded domain, not unique.

2.2. The quasi-geostrophic residual-mean equations

We now explicitly limit consideration to the quasi-geostrophic equations. A quantity θ is decomposed into a mean component $\overline{\theta}$ and an eddy component $\theta' = \theta - \overline{\theta}$.[2] The mean quasi-geostrophic momentum and buoyancy equations are then:

$$\partial_t \overline{\mathbf{u}_g} + \overline{\mathbf{u}_g} \cdot \nabla_H \overline{\mathbf{u}_g} + f_0 \hat{\mathbf{z}} \times \overline{\mathbf{u}_{ag}} + \beta y \hat{\mathbf{z}} \times \overline{\mathbf{u}_g} = -\frac{1}{\rho_0} \nabla_H \overline{p_{ag}} + \overline{\mathbf{S}} - \overline{\mathbf{u}_g' \cdot \nabla_H \mathbf{u}_g'}, \tag{3a}$$

$$\partial_t \overline{b} + \nabla_H \cdot \left(\overline{\mathbf{u}_g} \overline{b} \right) + \overline{w_{ag}} N_0^2 = \overline{B} - \nabla_H \cdot \overline{\mathbf{u}_g' b'}, \tag{3b}$$

where $\mathbf{u}_g$ is the geostrophic velocity, $\mathbf{u}_{ag}$ is the horizontal component of the ageostrophic velocity, and w_{ag} is the vertical component

[1] Here the horizontal skew-gradient is equivalent to a three-dimensional curl via $\hat{z} \times \nabla_H \Psi = -\nabla_H \times (\Psi \hat{\mathbf{z}}) = -\nabla \times (\Psi \hat{\mathbf{z}})$, where ∇ is the three-dimensional gradient operator.

[2] It is assumed that $\overline{(\ldots)}$ is a linear projection operator which commutes with the $\hat{\mathbf{z}} \times$ operator and with derivatives with respect to space and time. It is further assumed that f_0, βy, ρ_0, and N_0 have zero eddy component.