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A Boussinesq-scaled, pressure-Poisson water wave model

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ABSTRACT

Through the use of Boussinesq scaling we develop and test a model for resolving non-hydrostatic pressure profiles in nonlinear wave systems over varying bathymetry. A Green–Nagdhi type polynomial expansion is used to resolve the pressure profile along the vertical axis, this is then inserted into the pressure-Poisson equation, retaining terms up to a prescribed order and solved using a weighted residual approach. The model shows rapid convergence properties with increasing order of polynomial expansion which can be greatly improved through the application of asymptotic rearrangement. Models of Boussinesq scaling of the fully nonlinear $O(\mu^2)$ and weakly nonlinear $O(\mu^N)$ are presented, the analytical and numerical properties of $O(\mu^2)$ and $O(\mu^4)$ models are discussed. Optimal basis functions in the Green–Nagdhi expansion are determined through manipulation of the free-parameters which arise due to the Boussinesq scaling. The optimal $O(\mu^2)$ model has dispersion accuracy equivalent to a Padé [2,2] approximation with one extra free-parameter. The optimal $O(\mu^4)$ model obtains dispersion accuracy equivalent to a Padé [4,4] approximation with two free-parameters which can be used to optimize shoaling or nonlinear properties. In comparison to experimental results the $O(\mu^4)$ model shows excellent agreement to experimental data.

1. Introduction

Boussinesq type models have been used to make numerical predictions about nearshore wave phenomenon since their introduction by Peregrine (1967) and Madsen and Mei (1969). Through the use of a Boussinesq-type nondimensional scaling approach it was possible to design a relatively simple model which was applicable for shallow water waves, $\mu = kh \leqslant \pi/2$. Early models were weakly dispersive and weakly nonlinear, limiting their applicability in deeper water. Work by Madsen et al. (1991), Madsen and Srensen (1992) and Nwogu (1993) demonstrated that with careful manipulation of the equations it was possible to generate models with better dispersion characteristics, formally up to $O(\mu^4)$. Further work incorporated higher-order solutions with increased accuracy in shoaling and nonlinear interactions (Agnon et al., 1999; Kennedy et al., 2001; Schäffer and Madsen, 1995; Wei et al., 1995) and more relevant physics to the nearshore such as wave breaking and wave run-up (Chen et al., 2000; Kennedy et al., 2000; Lynett et al., 2002; Schäffer et al., 1993). The work of Gobbi and Kirby (2001), Gobbi et al. (2000) and Madsen and Schäffer (1998) extended the order of accuracy of the models, obtaining a dispersive relationship that was formally of $O(\mu^8)$, thereby expanding the range over which valid solutions could be obtained. Taking a different approach, Lynett and Liu (2004a,b) examined the increased accuracy gained through solutions to Boussinesq models over multiple vertical layers. The use of multiple layers provided an increase in the number of free-parameters which could be used to improve the model accuracy. Building on the advances made in Serre–Green–Nagdhi type modeling of water waves (Bonneton et al., 2011; Green and Naghdi, 1976; Serre, 1953), Zhang et al. (2013, 2014) were able to obtain an equivalent formal high-order of accuracy, even for low-order models, through the use of Green–Naghdi type polynomial expansions over the vertical domain for the velocities and the use of asymptotic rearrangement to find the optimal vertical basis functions.

Many of the higher-order models came at the cost of model complexity, which in many cases hindered the potential for these models to be adopted on a large scale. With the exception of Zhang et al. (2013), in order to obtain formally higher-order models it was necessary to use higher order spatial derivatives, which increases the computational cost. In addition mixed space/time derivatives are inherent in Boussinesq type models, see Peregrine (1967). Mixed

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space/time can be computationally expensive and difficult to implement, especially in two horizontal dimensions.

Wei and Kirby (1995) suggested the use of a higher-order Adams–Bashforth–Moulton predictor–corrector time integration scheme, thus ensuring that the dispersive error dominated the truncation error in the temporal discretization. Following their work and owing to the potential for greater spatial accuracy for lower computational cost hybrid spatial finite volume finite difference (Erduran et al., 2005; Shi et al., 2012; Gallerano et al., 2014) and finite element (Do Carmo et al., 1993; Li et al., 1999; Panda et al., 2014) implementations of the Boussinesq equations have been developed, these methods either employed higher-order predictor–corrector methods in time or took advantage of higher-order Runge–Kutta (RK) or Strong-Stability-Preserving-Runge–Kutta (SSPRK) methods in time. Furthermore, rearrangement of the dispersive terms such that the numerical solution of the one horizontal dimension model only requires the solution of a tridiagonal system addressed the computational cost issue associated with mixed space/time derivatives. However, in two horizontal dimensions it is still necessary to solve a set of two tightly coupled systems of equations.

Recently focus has shifted towards resolution of the dispersive terms through the solution of the Poisson type problems. This approach reduces the system to only one unknown, the pressure profile, in two-horizontal dimensions, as opposed to a set of two coupled velocity problems. A novel approach has been introduced by Antuono and Brocchini (2013) which focuses on solutions to a Poisson equation in the vertical velocity. This is achieved through a decomposition of the horizontal velocities into two parts, a depth-averaged velocity component and a deviation from depth-averaged term. The latter is further decomposed into rotational and irrotational components. Manipulation of the vorticity equations and the continuity equation yields a Poisson type problem, which when solved informs the deviation of the horizontal velocity from the depth-averaged component.

An approach that has recently gained traction is the resolution of dispersive effects by focusing on the non-hydrostatic pressure term. Building upon the concept introduced by Casulli and Stelling (1998), Casulli (1999) and Stansby and Zhou (1998) many highly accurate models have been developed and tested. Examples include the work of Yamazaki et al. (2009) on a depth integrated non-hydrostatic model, the Simulating WAves till SHore (SWASH) model developed by Stelling and Zijlema (2003), Zijlema and Stelling (2008) and Zijlema et al. (2011), the Non Hydrostatic WAVE (NHWAVE) model developed by Ma et al. (2012) and the CCHE2D model of Wei and Jia (2013). Each of these models have focused on solutions to the shallow water equations (SWE) or full Navier–Stokes equations where the non-hydrostatic pressure is treated as a single unknown which must be found numerically. The result is a model that does not include any mixed space/time derivatives, they do however involve an extra pressure-Poisson problem which must be solved to determine the non-hydrostatic pressure.

Zijlema and Stelling (2005) were able to demonstrate numerically that with an increased number of vertical layers it was possible to obtain high order accuracy for the dispersion characteristics. More recently Bai and Cheung (2013) derived the dispersion relationship and shoaling coefficient for the single and two layer models, as well as a more accurate hybrid single layer model. The results of the linear dispersion analysis for the single layer showed that it was only accurate for very shallow water waves. The more expensive two layer model demonstrated a significant improvement, equivalent to a Padé [2,4] approximation to the *Airy* solution, while the single layer hybrid model contained a free-parameter that could be used to optimize the dispersion to a Padé [2,2] approximation.

A separate approach based on an extension of the SWE's to include dispersive effects while retaining their hyberbolic structure has been proposed by Antuono et al. (2009). Assuming a sufficiently smooth bathymetry these so called Dispersive Nonlinear Shallow Water Equations (DNSWE's) are able to remain strictly hyperbolic through the inclusion of two pseudo-potential functions, thus they can take advantage of higher order finite volume or finite element numerical methods (Grosso et al., 2010).

The coupling of Boussinesq-type models with oceanographic models was stated as improvement of high urgency by Brocchini (2013) in his comprehensive analysis of the current state of Boussinesq models. The present work aims to take the advances made in classical Boussinesq theory and the recent work in multi-layer non hydrostatic pressure models to design a Boussinesq type model for the non-hydrostatic pressure which is suitable for a straightforward coupling with oceanographic models. Instead of vertical layers, a Green–Nagdhi type polynomial expansion is used to resolve the pressure over the vertical domain. The result is a simple model for the non-hydrostatic pressure, which is found through the solution to the pressure-Poisson equation and enforcement of the bottom boundary condition on pressure. The combination of a Green–Nagdhi type polynomial expansion with Boussinesq-type scaling provides free-parameters, which can be manipulated using the principles of asymptotic rearrangement to optimize for various properties including dispersion, shoaling and nonlinear interactions. At lowest-order the model is comparable to standard Boussinesq models as well as the hybrid single layer model of Bai and Cheung (2013) At higher-order the model compares well with the higher order models of Gobbi and Kirby (1999) and Zhang et al. (2013), but is relatively easier to implement and does not contain higher-order spatial derivatives or mixed space/time derivatives.

This paper is organized as follows: Section 2 introduces the dimensionless scaling and the governing equations for the model. In Section 3 a pressure-Poisson model using Boussinesq scaling and Green–Nagdhi type expansions in the vertical axis is developed. Section 4 discusses the analytical properties of the model and compares them with well known analytical results. Section 5 discusses several validation experiments conducted using a numerical solution to the model. Finally the conclusions of the paper are given in Section 6. Appendix A provides details of reduction in the degrees of freedom for higher order solutions, appendix B provides details of the linear dispersion and shoaling analysis and appendix C provides details of the nonlinear analysis for the second order nonlinear term.

2. Scaling

For the present study we will consider flow of a constant density inviscid fluid without bottom or surface shear stresses. We employ a Cartesian coordinate system (x^*, y^*, z^*), where z^* represents the vertical axis centred on the still-water plane pointing upwards. The full vertical profile stretches from the bottom bathymetry at $z^* = -h^*(x^*, y^*)$ to the free-surface $z^* = \eta^*(x^*, y^*, t^*)$. The following nondimensional quantities are defined:

$$
\begin{aligned}
&(x, y) = k_0(x^*, y^*), \quad (u, v) = \frac{h_0}{a_0\sqrt{g_0 h_0}}(u^*, v^*),\\
&\eta = \frac{\eta^*}{a_0}, \quad z = \frac{z^*}{h_0},\\
&w = \frac{w^*}{a_0 k_0\sqrt{g_0 h_0}}, \quad P = \frac{P^*}{\rho g_0 a_0},\\
&h = \frac{h^*}{h_0}, \quad g = \frac{g^*}{g_0},\\
&t = k_0\sqrt{g_0 h_0}\,t^*,
\end{aligned} \tag{1}
$$