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A numerical method for the two layer shallow water equations with dry states

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ABSTRACT

A numerical method is proposed for solving the two layer shallow water equations with variable bathymetry in one dimension based on high-resolution f-wave-propagation finite volume methods. The method splits the jump in the fluxes and source terms into waves propagating away from each grid cell interface. It addresses the required determination of the system's eigenstructure and a scheme for evaluating the flux and source terms. It also handles dry states in the system where the bottom layer depth becomes zero, utilizing existing methods for the single layer solution and handling single layer dry states that can exist independently. Sample results are shown illustrating the method and its handling of dry states including an idealized ocean setting.

1. Introduction

The multilayer shallow water equations have come under increasing interest as a model for primarily long wave phenomena where vertical structure either plays an important role in the flow or the internal structure itself is of interest. The primary barrier to the use of these equations more broadly has been the complexity and computational cost of the required solvers. Past approaches to this problem have included the use of more diffusive solvers (Salmon, 2002), relaxation approaches (Abgrall and Karni, 2009), and decoupling schemes that obey entropy laws (Bouchut and Morales de Luna, 2008). Another difficulty is the loss of hyperbolicity commonly associated with the physical mechanism of Kelvin–Helmholtz instabilities. Work has also been done to mitigate this effect by applying physically motivated momentum transfer to stabilize the system (Castro-Díaz et al., 2010).

To address some of these difficulties, the numerical method presented here attempts to overcome many of the drawbacks of previous methods while remaining accurate in many scenarios of interest. In particular this method is intended for oceanic applications where a two-layer model could be used to represent a relatively shallow top "boundary" layer and a deeper "abyssal" layer. This implies that the bottom layer can be assumed to become dry before the top layer. Forcing physics with limited vertical extent such as wind or friction drag are of particular interest, such as in storm surge applications. In these cases issues such as hyperbolicity are not a concern due to the regime scales being considered. On the other hand, wetting and drying of the internal surface will occur along the shelf break and must be handled carefully. Although the step to multiple layers only introduces a limited representation of the three dimensional nature of many of these flows, it is computationally less-expensive than a fully three-dimensional simulation, making the required resolution of some oceanic flows attainable even on modest computing hardware.

The presentation of the numerical method begins with an introduction in Section 2 to the salient features of the multilayer shallow water equations including methods for evaluating the eigenspace. A discussion of the basic components of the f-wave approach follows with the specific implementation details for the multilayer shallow water equations and dry states in Section 3. Finally, example solutions from the numerical method are presented for test cases where dry states are involved in Section 4. It should be noted that the presentation will be restricted to the one-dimensional multilayer shallow water equations for clarity. Many of the salient issues are present in one-dimension and the extension to two-dimensions follows directly from the methods presented here with the formulation of a transverse Riemann solver and extra care dealing with dry-states in the transverse directions. This topic is left to be presented in future work.

2. Multilayer shallow water equations

The multilayer shallow water equations can be derived by integrating the Euler equations in the vertical coordinate direction as in the case of the single-layer equations. The difference between

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the multilayer and single-layer shallow water equations is the addition of vertical variation in the density and velocity. In one-dimension and for two layers the equations are often written as

$$\begin{aligned}
&(h_1)_t + (h_1 u_1)_x = 0,\\
&(h_1 u_1)_t + \left(h_1 u_1^2 + \frac{1}{2} g h_1^2\right)_x = -g h_1 (h_2 + b)_x,\\
&(h_2)_t + (h_2 u_2)_x = 0,\\
&(h_2 u_2)_t + \left(h_2 u_2^2 + \frac{1}{2} g h_2^2\right)_x = -r g h_2 (h_1)_x - g h_2 b_x
\end{aligned} \tag{1}$$

where h_i and u_i are the depths and velocities in each layer respectively, b is the bathymetry from a reference sea-level, g the gravitational acceleration, and $r \equiv \rho_1/\rho_2$ the ratio of the layer densities (see Fig. 1). Note that we have enumerated the layers with the top layer being indexed first. The result of the vertical integration and hydrostatic assumption is a system of partial differential equations resembling two sets of single-layer shallow water equations with the addition of a coupling term between the layers. It is important to note that this coupling is due solely to the integration of the hydrostatic pressure and does not represent momentum transfer due to drag between the layers.

Another form of Eqs. (1) involves forgoing the division of the equations by the density of each layer in the derivation and integrating the bottom layer coupling term so that a term appears in the flux of the second-layer's momentum equation rather than as a source term (Abgrall and Karni, 2009). In this case, the non-conservative coupling terms in each layer are symmetric. Using the same notation as before, these equations can be written as

$$\begin{aligned}
&(\rho_1 h_1)_t + (\rho_1 h_1 u_1)_x = 0,\\
&(\rho_1 h_1 u_1)_t + \left(\rho_1 h_1 u_1^2 + \frac{1}{2} g \rho_1 h_1^2\right)_x = -g \rho_1 h_1 (h_2)_x - g \rho_1 h_1 b_x,\\
&(\rho_2 h_2)_t + (\rho_2 h_2 u_2)_x = 0,\\
&(\rho_2 h_2 u_2)_t + \left(\rho_2 h_2 u_2^2 + \frac{1}{2} g \rho_2 h_2^2 + g \rho_1 h_2 h_1\right)_x = g \rho_1 h_1 (h_2)_x - g \rho_2 h_2 b_x.
\end{aligned} \tag{2}$$

The symmetry in the non-conservative products has the benefit that the transfer of momentum due to these coupling terms moves directly between the layers which will be advantageous numerically. For the remainder of the discussion we will focus our attention solely on the two-layer case solving the system of equations in (2). Extensions of these methods to greater numbers of layers are possible, but for simplicity these complications will be ignored in the rest of the current discussion and is left as future work on the subject.

Since the intention is to consider oceanic applications, an important simplification of the nonlinear equations is the linearization about an ocean at rest. Taking the steady state where $\hat{u}_1$ and $\hat{u}_2$ are zero and the sea surface $\hat{\eta}_1$ and internal surface $\hat{\eta}_2$ are constant we can rewrite (2) as

$$\begin{aligned}
&(\tilde{h}_1)_t + (\tilde{\mu}_1)_x = 0,\\
&(\tilde{\mu}_1)_t + g \hat{h}_1 (\tilde{h}_1 + \tilde{h}_2)_x = 0,\\
&(\tilde{h}_2)_t + (\tilde{\mu}_2)_x = 0, \text{ and}\\
&(\tilde{\mu}_2)_t + g \hat{h}_2 \left[(\tilde{h}_2)_x + r(\tilde{h}_1)_x\right] = 0.
\end{aligned} \tag{3}$$

where we have defined $\hat{h}_1 = \hat{\eta}_1 - \hat{\eta}_2, \tilde{h}_1 = \tilde{\eta}_1 - \tilde{\eta}_2, \hat{h}_2 = \hat{\eta}_2 - b$, and $\tilde{h}_2 = \tilde{\eta}_2$ for convenience and $\tilde{\mu}_i$ is the perturbation to the momentum of the background ocean at rest such that $\tilde{\mu}_i = \hat{h}_i \tilde{u}_i$ (see Fig. 1). Note with these definitions $\hat{h}_2$ is spatially dependent due to the inclusion of b.

With these equations, we can write the system (3) in the form $\tilde{q}_t + A(\hat{q}) \tilde{q}_x = 0$, where

$$\hat{q} = \begin{bmatrix} \hat{h}_1 \\ 0 \\ \hat{h}_2 \\ 0 \end{bmatrix}, \quad \tilde{q} = \begin{bmatrix} \tilde{h}_1 \\ \hat{h}_1 \tilde{u}_1 \\ \tilde{h}_2 \\ \hat{h}_2 \tilde{u}_2 \end{bmatrix}, \quad \text{and } \tilde{A}(\hat{q}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ g\hat{h}_1 & 0 & g\hat{h}_1 & 0 \\ 0 & 0 & 0 & 1 \\ rg\hat{h}_2 & 0 & g\hat{h}_2 & 0 \end{bmatrix}.$$

2.1. Eigenspace

The eigenspace of hyperbolic PDEs is often of interest and one of the primary sources of difficulties when considering the multilayer shallow water equations. If one were to directly use the flux Jacobian of (2) the eigenvalues and eigenvectors would be identical to two uncoupled shallow water equation systems. This approach of using a splitting of the layers was shown to be unstable (Castro et al., 2001) unless suitable corrections are used (Bouchut and Morales de Luna, 2008). Since the wave speeds predicted by this approach do not take into account the coupling between the layers, this approach is not desirable for methods that depend on this information to construct a Riemann solution. Instead it is common to write the system in a quasi-linear form $q_t + \tilde{A}(q) q_x = \tilde{S}(q)$ where

$$q = [\rho_1 h_1, \rho_1 h_1 u_1, \rho_2 h_2, \rho_2 h_2 u_2]^T$$

and

$$\tilde{A}(q) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ gh_1 - u_1^2 & 2u_1 & rgh_1 & 0 \\ 0 & 0 & 0 & 1 \\ gh_2 & 0 & gh_2 - u_2^2 & 2u_2 \end{bmatrix} \text{ and } \tilde{S}(q) = \begin{bmatrix} 0 \\ -g\rho_1 h_1 b_x \\ 0 \\ -g\rho_2 h_2 b_x \end{bmatrix}. \tag{4}$$

The characteristic polynomial of the matrix $\tilde{A}(q)$ is then

$$((\lambda - u_1)^2 - gh_1)((\lambda - u_2)^2 - gh_2) - rg^2 h_1 h_2 = 0. \tag{5}$$

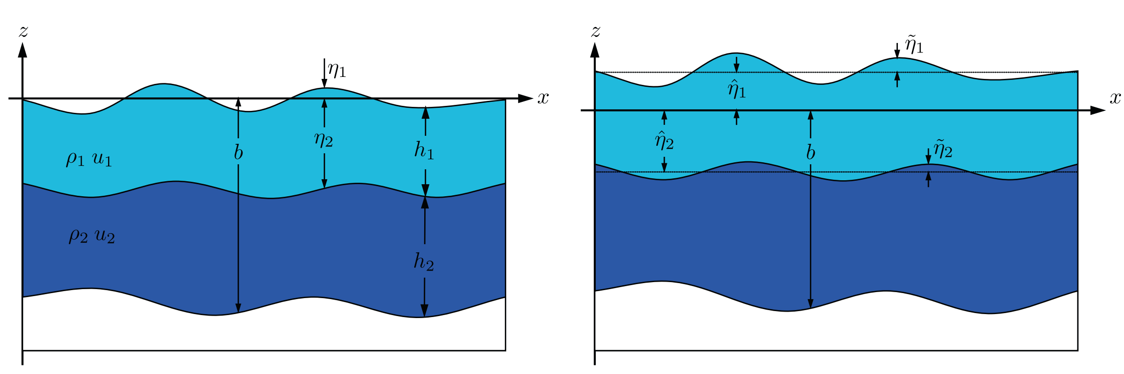

Fig. 1. Coordinates for a one-dimensional system with two-layers and varying bathymetry in general and for the linearized case.