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The stochastic runoff-runon process: Extending its analysis to a finite hillslope

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ABSTRACT

The stochastic runoff-runon process models the volume of infiltration excess runoff from a hillslope via the overland flow path. Spatial variability is represented in the model by the spatial distribution of rainfall and infiltration, and their "correlation scale", that is, the scale at which the spatial correlation of rainfall and infiltration become negligible. Notably, the process can produce runoff even when the mean rainfall rate is less than the mean infiltration rate, and it displays a gradual increase in net runoff as the rainfall rate increases.

In this paper we present a number of contributions to the analysis of the stochastic runoff-runon process. Firstly we illustrate the suitability of the process by fitting it to experimental data. Next we extend previous asymptotic analyses to include the cases where the mean rainfall rate equals or exceeds the mean infiltration rate, and then use Monte Carlo simulation to explore the range of parameters for which the asymptotic limit gives a good approximation on finite hillslopes. Finally we use this to obtain an equation for the mean net runoff, consistent with our asymptotic results but providing an excellent approximation for finite hillslopes. Our function uses a single parameter to capture spatial variability, and varying this parameter gives us a family of curves which interpolate between known upper and lower bounds for the mean net runoff.

1. Introduction

The volume of catchment discharge that reaches a stream via the overland flow path is critical for water quality prediction, because it is via this pathway that most particulate pollutants are generated and transported to the stream channel, via surface erosion processes. Two of the key properties determining this volume are the rainfall rate and the infiltration rate. In natural systems both these rates are variable in both space and in time.

Suppose that our hillslope is divided into cells. If the rainfall rate exceeds the infiltration rate in a given cell, then the excess will flow overland to the next cell downhill. Thus the water flowing into a cell is given by the sum of the rainfall and runon from the cell above. Any excess, after infiltration is taken into account, becomes runoff. The resulting system is highly non-linear, because runoff is truncated below at zero. Nahar (2003) showed that for soils with moderate to high mean saturated conductivity relative to rainfall rate, the runoff-runon process plays an important part in determining the total overland discharge for a hillslope. These conditions are typical in temperate forests, where saturated conductivity values are usually high, and are common in many other landscapes for the majority of rainfall events (Dunkerley, 2008).

Because of the complexity of the problem, models that incorporate both spatial and temporal variability have, to date, been analysed using numerical simulation methods. Our interest is in analytic solutions. The most common simplification made in this context is to neglect spatial variability and model rainfall and infiltration as a function of time only. This can be attributed to the early development of analytical expressions for the temporal change in infiltration rate at a point (Green and Ampt, 1911). For catchment scale predictions these point-scale results have generally been scaled up by optimizing the infiltration parameters using catchment or hillslope runoff time-series data. As a result of this scaling process, the parameters lose their physical meaning (e.g. see discussion by Grayson et al., 1992). A recent alternative is the stochastic runoff-runon process introduced by Jones et al. (2009) and developed in Jones et al. (2013) and Harel and Mouche (2013, 2014). The stochastic runoff-runon process allows for spatial variability but assumes temporal stationarity. It does however admit analytic asymptotic solutions, with parameters that retain their physical meaning. In this paper we pay particular attention

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to how the model behaves on finite hillslopes, when the previously obtained asymptotic solutions are not available.

Hawkins and Cundy (1987) were the first to propose an analytic solution to the net runoff generation problem incorporating variability in the spatial dimension. Hawkins and Cundy showed that for an area with constant rainfall and spatially variable infiltration there exist maximum and minimum curves relating the net runoff rate to the precipitation rate; see Fig. 1. The curves are derived by arranging the point infiltration values from largest to smallest, or vice versa. The true (but generally unknown) function relating precipitation rate to net runoff rate must lie between these enveloping curves. Assuming that the distribution of infiltration rates has an exponential density with mean m_i and that the rainfall is constant with rate m_p (both in mm/h), the minimum net runoff rate is $m_p - m_i$, for $m_p \geqslant m_i$, and the maximum is

$$m_p - m_i(1 - e^{-m_p/m_i}). \tag{1}$$

Note that the function depends on precipitation rate and not time, as the system is assumed to be in temporal equilibrium. We also see that runoff is generated even when the precipitation rate is lower than the average infiltration rate, and it increases gradually as the precipitation rate increases. These are characteristic consequences of including spatial variability.

The Hawkins and Cundy model has not received widespread attention, despite the fact that Yu and others have reported considerable success using the maximum net runoff curve as the basis of a runoff model at the plot scale (Yu et al., 1997, 1998; Yu, 1999; Fentie et al., 2002; Kandel et al., 2005). This approach was found to perform better than the time-variant, space-invariant Green and Ampt (1911) model for the prediction of infiltration excess runoff at the plot scale (Yu, 1999). One of the main contributions of the present paper is the derivation of a family of curves for the mean net runoff, which smoothly interpolate between the upper and lower bounds of Hawkins and Cundy; see Section 5.1, Eq. (19).

As a consequence of modelling runoff from each cell, the stochastic runoff-runon process can be used to analyse hillslope connectivity, whereby adjacent cells are connected if there is runoff from one to the other. In this context the process has been called the stochastic runoff connectivity (SRC) process (Sheridan et al., 2009a,b). Harel and Mouche (2014) also consider connectivity through the lens of the stochastic runoff-runon process, extending the model to include lateral diffusion of runoff. We do not consider connectivity explicitly in this paper, though note that in Section 3.2 we do draw conclusions about the proportion of the hillslope that contributes to net runoff.

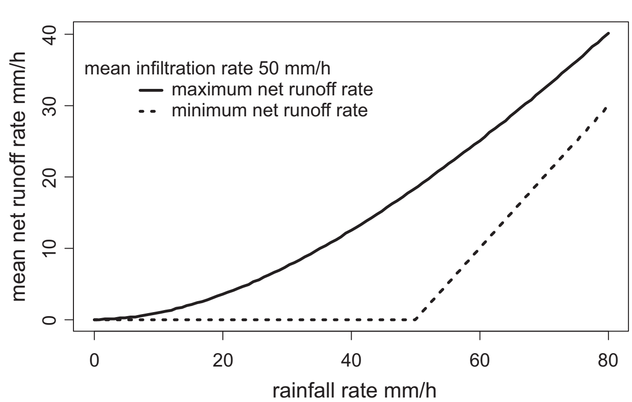

Fig. 1. Bounds on the relationship between precipitation rate (mm/h) and net runoff rate (mm/h), shown for the case of a mean infiltration rate of 50 mm/h. Modified from Hawkins and Cundy (1987).

The structure of the paper is as follows. In Section 2 we give a definition of the (time-stationary one-dimensional) stochastic runoff-runon process, and then fit it to some experimental data using maximum likelihood. The fit is quite good, lending credence to the model.

In Section 3 we present an analysis of the asymptotic properties of the model, as the length of the hillslope tends to infinity. Three regimes emerge, depending on whether the mean rainfall rate is less than, equal to, or greater than the mean infiltration rate. We refer to these regimes as subcritical, critical and supercritical respectively. Then in Section 4 we use Monte Carlo simulation to quantify the scales at which asymptotic results can be used to approximate runoff behaviour on finite hillslopes.

In Section 5 we develop a function for the mean runoff rate on a finite hillslope, as a function of the mean rainfall rate, which spans all three regimes, subcritical, critical and supercritical. We compare our function for the mean runoff rate to that of Hawkins and Cundy, and show that as you increase the spatial variability of the infiltration rate, our function smoothly transitions from their lower bound to their upper bound.

A discussion and summary of our results are given in Sections 6 and 7.

2. The stochastic runoff-runon process

The stochastic runoff-runon process is a stochastic time-invariant model for the flow of infiltration-excess runoff down a planar hillslope. We model the hillslope as a series of parallel and independent strips perpendicular to the bottom edge. Each strip is broken up into a line of blocks or cells of equal size, and we suppose that within each block the rate of rainfall and infiltration are fixed. The flow of runoff from one cell to the next downslope can be considered a stochastic process *spatially* indexed by the position of the cell down the strip.

The runoff-runon model is constructed in two steps. Firstly we consider the runoff generation down a *single* strip of land from the top to the bottom of the hillslope, perpendicular to the contours, with a random arrangement of rainfall and infiltration capacity down its length. Analysis of this component of the model draws on queuing theory. Next, we consider the properties of the aggregated output from these strips. Analysis of this component of the model depends on the central limit theorem. We can use the central limit theorem because we assume that runoff is confined within strips, so that the net runoff from individual strips is independent. This follows from our assumption that the hillslope is "planar", that is, the contours are parallel.

2.1. Single strip runoff-runon model

We consider a single strip of land, width l_x, divided into blocks of length l_y. Number the blocks $1, 2, \ldots, n$, starting at the top of the slope. Let X_k be the rate at which water runs from block k to block $k+1$, that is, the flow from block k to $k+1$, in $\mathrm{m}^3\,\mathrm{h}^{-1}$. Let p_k be the precipitation (rainfall) rate and i_k the infiltration rate for block k, (both are fluxes, measured in $\mathrm{mm\,h}^{-1}$), assumed to be constant over time. If $p_k > i_k$ then on average runoff builds up down the length of the block, conversely if $p_k < i_k$ then on average the runoff declines. Let the depth of water at the end of block k be d_k and let its speed be v_k, then the volume of water leaving the block per unit time is $l_x d_k v_k = X_k$. If v_k were constant, then we would have $d_k \propto l_y$. Let $P_k = l_x l_y p_k/1000$ be the flow of rain falling on block k, and let $I_k = l_x l_y i_k/1000$ be the maximum flow of water absorbed by block k (both in $\mathrm{m}^3\,\mathrm{h}^{-1}$). Here we assume P_k represents incident rainfall if there is no canopy or over-story, or through-fall if there is an over-storey. If we assume that there is *no significant runoff onto*