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Approximate solutions for Forchheimer flow during water injection and water production in an unconfined aquifer

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SUMMARY

Understanding the hydraulics around injection and production wells in unconfined aquifers associated with rainwater and reclaimed water aquifer storage schemes is an issue of increasing importance. Much work has been done previously to understand the mathematics associated with Darcy's law in this context. However, groundwater flow velocities around injection and production wells are likely to be sufficiently large such as to induce significant non-Darcy effects. This article presents a mathematical analysis to look at Forchheimer's equation in the context of water injection and water production in unconfined aquifers. Three different approximate solutions are derived using quasi-steady-state assumptions and the method of matched asymptotic expansion. The resulting approximate solutions are shown to be accurate for a wide range of practical scenarios by comparison with a finite difference solution to the full problem of concern. The approximate solutions have led to an improved understanding of the flow dynamics. They can also be used as verification tools for future numerical models in this context.

1. Introduction

With the ever increasing significance of climate change induced rainfall variability combined with increasing urban populations, understanding the well hydraulics associated with managed aquifer recharge schemes continues to be an important research topic for water managers around the world (Bouwer, 2002; Dillon, 2005; Sheng, 2005; Pliakas et al., 2005). Such schemes typically involve storing rainwater in aquifers during abundant periods and extracting it when droughts occur (Donovan et al., 2002; Khan et al., 2008). In some cases, reclaimed wastewater is injected into aquifers with a view that aquifer storage can provide additional treatment (Bouwer, 2002; Dillon, 2005) such that, after sufficient time, the water satisfies local drinking water quality standards (Rygaard et al., 2011).

Appropriate hydraulic models can serve to estimate the conditions under which overflow induced by well recharge might occur (Sheng, 2005), to estimate the recovery potential of stored water, to estimate resident times in aquifers for bioremediation capacity, to forecast negative impacts of recharge on building foundations, pipelines and deep rooted vegetation and to compute energy requirements for aquifer recharge recovery schemes.

In most studies of well hydraulics, it is assumed that the flow behavior can be described by Darcy's law. By further taking into account the continuity equation, the water table evolution in unconfined aquifers can be described by a single non-linear partial differential equation (PDE), the Boussinesq equation (e.g. Bear, 1979).

Existing analytical solutions of the non-linear Boussinesq equation for radial, transient, unconfined flow induced by water injection to an unconfined aquifer are limited to Darcy-flow conditions and to initially dry aquifer conditions (Yeh and Chang, 2013). Babu and van Genuchten (1980) used similarity transforms to transform the Boussinesq equation to an ordinary differential equation (ODE) and then provided an approximate solution using a perturbation expansion. A similar ODE was derived using similarity transforms by Barenblatt et al. (1990), to which Li et al. (2005) provided asymptotic solutions for both small and large values of the similarity variable. Li et al. (2005) combined these expansions to yield an approximate solution valid for all values of the similarity variable, which they verified by comparison to equivalent numerical results.

Analytical solutions of the linearized radial or two-dimensional Boussinesq equation for transient flow induced by water injection to an unconfined aquifer are more abundant (Hunt, 1971; Marino

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and Yeh, 1972; Rai and Singh, 1995; Manglik et al., 1997; Teloglou et al., 2008). Both the cases that water is introduced to an aquifer by an injection well (Marino and Yeh, 1972), or by a recharge basin (Rai et al., 1998) are examined. A linearization of the Boussinesq equation either in terms of h, (Rai and Singh, 1995) or in h^2, (where h is the water table elevation relative to the base of the aquifer), is generally adopted. The resulting linear PDE is solved using the Laplace transform method, the finite Hankel transform approach and/or the eigenvalue–eigenfunction method (Marino and Yeh, 1972; Teloglou et al., 2008; Rai et al., 1998). The application range of the solutions above is limited to the case that the perturbation of the water table elevation induced by the water recharge is small.

Due to high velocities, inertial non-Darcy flow conditions may occur in the well vicinity (Mathias and Todman, 2010; Moutsopoulos et al., 2009). Non-Darcy effects cause additional head losses, so that for the injection well problem, the rise of head at the near well field would be higher than predicted by Darcy's law. The potential engineering implications of these non-Darcy effects are increased danger of overflow for water injection and increased energy consumption for water production.

Semi-analytical solutions for one-dimensional (non-radial) transient Forcheimer flow in unconfined aquifers have previously been developed by Bordier and Zimmer (2000) and Moutsopoulos (2007, 2009). A semi-analytical solution for one-dimensional steady state radial flow in unconfined aquifers has previously been presented by Terzidis (2003). However, to better understand the role of non-Darcy effects during water injection in unconfined aquifers, we present a series of new approximate analytical solutions to explore one-dimensional transient radial Forchheimer flow in unconfined aquifers.

The outline of this article is as follows: The governing equations for transient one-dimensional radial Forchheimer flow in a homogenous and isotropic unconfined aquifer are presented. The equations are normalized using an appropriate set of dimensionless transformations. Following the ideas of Bordier and Zimmer (2000) and Sen (1986), two different approximate solutions for Darcian flow and strongly non-Darcian flow are derived for initial saturated zones of arbitrary thickness by invoking a quasi-steady-state assumption. Following Mathias et al. (2008), an approximate solution for non-Darcy flow in an aquifer with a moderately deep initial saturated zone is derived using the method of matched asymptotic expansion. The performance of the new approximate solutions are verified by comparison to a finite difference solution of the full problem.

2. Governing equations

Consider the injection/production of water into/from a homogenous and isotropic unconfined aquifer. Considering the so-called Dupuit assumption (that vertical flow is negligible) (Bear, 1979), an appropriate one-dimensional mass conservation equation can be written as

$$S_y \frac{\partial h}{\partial t} = -\frac{1}{r}\frac{\partial (rhq)}{\partial r} \tag{1}$$

where (Forchheimer, 1901)

$$q + \frac{bK}{g}|q|q = -K\frac{\partial h}{\partial r} \tag{2}$$

and S_y [–] is the specific yield, h [L] is the water table elevation above a horizontal impermeable formation, t [T] is time, r [L] is radial distance from an injection well, b [L^{-1}] is the Forchheimer coefficient, K [LT^{-1}] is the hydraulic conductivity of the unconfined aquifer and g [LT^{-2}] is the gravitational acceleration constant.

The relevant initial and boundary conditions can be stated as:

$$\begin{array}{lll} h = h_i, & r > 0, & t = 0 \\ 2\pi rhq = \gamma Q_0, & r \to 0, & t > 0 \\ q = 0, & r \to \infty, & t > 0 \end{array} \tag{3}$$

where h_i [L] is a uniform initial water table elevation, Q_0 [L^{3}T^{-1}] is a positive valued flow rate associated with a production well or injection well located at $r = 0$ with $\gamma = 1$ for an injection well and $\gamma = -1$ for a production well.

Note that Eq. (2) can rearranged to the form (Mathias et al., 2014; Mathias and Wen, 2015)

$$q = -FK\frac{\partial h}{\partial r} \tag{4}$$

where

$$F = 2\left[1 + \left(1 + \frac{4bK^2}{g}\left|\frac{\partial h}{\partial r}\right|\right)^{1/2}\right]^{-1} \tag{5}$$

3. Dimensionless transformation

It is helpful at this stage to apply the following dimensionless transformations:

$$t_D = \frac{Kt}{S_y H}, \quad r_D = \frac{r}{H}, \quad h_D = \frac{h - h_i}{H}, \quad q_D = \frac{q}{K}, \quad \epsilon = \frac{h_i}{H}, \quad \beta = \frac{bK^2}{g} \tag{6}$$

where

$$H = \left(\frac{Q_0}{2\pi K}\right)^{1/2} \tag{7}$$

such that the above problem reduces to

$$\frac{\partial h_D}{\partial t_D} = -\frac{1}{r_D}\frac{\partial}{\partial r_D}[r_D(h_D + \epsilon)q_D] \tag{8}$$

$$q_D = -F\frac{\partial h_D}{\partial r_D} \tag{9}$$

$$F = 2\left[1 + \left(1 + 4\beta\left|\frac{\partial h_D}{\partial r_D}\right|\right)^{1/2}\right]^{-1} \tag{10}$$

$$\begin{array}{lll} h_D = 0, & r_D > 0, & t_D = 0 \\ r_D(h_D + \epsilon)q_D = \gamma, & r_D \to 0, & t_D > 0 \\ q_D = 0, & r_D \to \infty, & t_D > 0 \end{array} \tag{11}$$

Note that it is also possible to state that

$$q_D + \beta|q_D|q_D = -\frac{\partial h_D}{\partial r_D} \tag{12}$$

4. Analytical solution for large ϵ and zero β

The case of very large ϵ corresponds to the case of very large values of the initial water table elevation or very small values of the flow-rate, such that either the raise in water table elevation induced by water injection or the drawdown induced by water extraction can be assumed negligible. In this way, the cross-sectional area, through which groundwater flow takes place, can be assumed uniform and constant, such that flow processes can be described by the same equations ordinarily used to describe confined aquifers. The case of zero β corresponds to a problem