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Finite analytic method based on mixed-form Richards' equation for simulating water flow in vadose zone

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SUMMARY

In this paper, we develop a finite analytic method (FAMM), which combines flexibility of numerical methods and advantages of analytical solutions, to solve the mixed-form Richards' equation. This new approach minimizes mass balance errors and truncation errors associated with most numerical approaches. We use numerical experiments to demonstrate that FAMM can obtain more accurate numerical solutions and control the global mass balance better than modified Picard finite difference method (MPFD) as compared with analytical solutions. In addition, FAMM is superior to the finite analytic method based on head-based Richards' equation (FAMH). Besides, FAMM solutions are compared to analytical solutions for wetting and drying processes in Brindabella Silty Clay Loam and Yolo Light Clay soils. Finally, we demonstrate that FAMM yields comparable results with those from MPFD and Hydrus-1D for simulating infiltration into other different soils under wet and dry conditions. These numerical experiments further confirm the fact that as long as a hydraulic constitutive model captures general behaviors of other models, it can be used to yield flow fields comparable to those based on other models.

1. Introduction

Understanding of groundwater recharge, evaporation, and the mechanisms controlling movement of moisture and pollutants in the vadose zone to groundwater reservoirs are of great importance in environmental and agricultural engineering fields (Jing et al., 2014; Wang et al., 2011, 2014; Wang et al., 2016). Quantitative investigations of these processes and mechanisms often rely on Richards' equation. Because Richards' equation is nonlinear, analytical solutions are tractable only under special cases. Therefore, numerical models are deemed to be more appropriate tools for dealing with any realistic field problems. According to the types of variables used, Richards' equation generally can be classified into three types: water content (θ)-based form, pressure head (h)-based form, and a mixed form. Crevoisier et al. (2009) reported that different forms of Richards' equation could have significantly influenced computational efficiency, accuracy and behavior of the solutions. The h-based Richards' equation is more versatile because it can be applied to variably saturated flow in heterogeneous media (e.g., Brunone et al., 2003, and Mao et al., 2011; Yeh et al., 2015a). However, Milly (1985) and Celia et al. (1990) reported that the h-based Richards' equation is difficult to solve using numerical approaches. Numerical solutions to this equation may yield results involving large mass balance errors, unless small grids and fine time steps are employed. In order to avoid these problems, many researchers have suggested that the mixed-form Richards' equation, which can maintain the conservative property with less computational efforts, should be adopted to simulate water flow in unsaturated zone (Allen and Murphy, 1986; Celia et al., 1987, 1990).

Celia et al. (1990) developed a modified Picard finite difference (MPFD) method for the mixed-form Richards' equation and showed that it yielded robust and reliable numerical solutions for unsaturated flow problems. They stated that the mixed-form equation combines advantages in the h-based and the θ-based equations while circumventing difficulties associated with each one. Their numerical solutions to the mixed-form Richards' equation, nevertheless, are subject to truncation error as the most numerical approaches.

Abbreviations: FAMM, finite analytic method based on mixed-form Richards' equation; MPFD, modified Picard finite difference approximation; FAMH, finite analytic method based on head-based Richards' equation.

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A finite analytic method (FAM) was presented by Chen and Chen (1981, 1984) to solve heat conduction, and Navier-Strokes equations. Zeng and Li (1987) advocated that the finite analytic method could minimize the truncation errors and yield stable numerical solutions. Hwang et al. (1985) applied FAM to solving two-dimensional solute transport equation, and reported that FAM produced accurate results and eliminated numerical dispersion for large Peclet numbers. A hybrid Laplace transform finite analytic method (LTFAM) was developed by Wang et al. (2012) for solving advection–dispersion equations. Comparing results of LTFAM with those of the analytical solutions, they concluded that the LTFAM generated highly accurate numerical solutions even under the conditions where Peclet numbers are greater than 50. Using an optimal time-weighting factor, Tsai et al. (1993) built an FAM for solving the h-based Richards' equation with irregular boundaries. By combining with fine-point local elements and nine-point local elements, they reported that the FAM could easily incorporate irregular surface boundary conditions. However, they found that FAM, based on the h-based form of Richards' equation, cannot guarantee global mass conservation. Recently, Zhang et al. (2015) formulated an FAMH, which is based on Kirchhoff transform of the h-based Richards' equation, and showed that FAMH can lead to relatively high accurate and stabile numerical solutions. Furthermore, they proved the convergence and stability of the FAMH by a rigorous mathematical analysis.

To our knowledge, no study has applied the FAM to the mixed-form Richards' equation up to date. The purpose of this study thus is to develop a FAM for the mixed-form Richards' equation (i.e., FAMM) and to investigate its mass conservative property and accuracy. We first present a finite analytic computational framework of the mixed-form Richards' equation. Numerical experiments are then presented that evaluate performance of this new method. They include those for comparison solutions of FAMM and MPFD with analytical solutions, derived for two constitutive relationship models, for transient soil water pressure distributions. At last, FAMM solutions are compared with those from MPFD and Hydrus-1D models, which are formulated with Mualem-van Gehuchten constitutive relationship model for transient infiltration into different types of soils under different initial conditions.

2. Formulation of FAMM

This study assumes that moisture movement in the unsaturated zone is described by following equation (Brunone et al., 2003).

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left[\bar{k}(h)\frac{\partial h}{\partial z}\right] - \frac{\partial \bar{k}(h)}{\partial z} \tag{1}$$

in which θ is moisture content (cm^3/cm^3), h is soil water pressure head (cm), z denotes the vertical dimension, and downward direction is positive (cm), $\bar{k}(h)$ represents the unsaturated hydraulic conductivity of the soil (cm/h), which is a function of soil water pressure head, and t is time (h).

Generally speaking, two types of constitutive mathematical models for describing hydraulic properties of soils under unsaturated conditions have been widely used by soil scientists. One is the Gardner model (1958) (hereafter Type 1 model), in which

$$\bar{k} = k_s e^{(\beta h)} \tag{2}$$

for hydraulic conductivity and pressure head relationships, and

$$\theta = \theta_r + (\theta_s - \theta_r)e^{(\beta h)} \tag{3}$$

for moisture and pressure head relationships. These two relationships lead to

$$\frac{d\theta}{d\bar{k}} = \frac{(\theta_s - \theta_r)}{k_s} \tag{4}$$

Notice that Eq. (4) is a constant, independent of pressure head. This model allows Srivastava and Yeh (1991) to derive an analytical solution for solving vertical infiltration problems. This exponential model has been popular because of its simplicity and convenience for mathematical analyses. It, nevertheless, fits the observed $K(h)$ or $\theta(h)$ data in an approximate sense.

Another mode is called Type 2 model, which is a combination of the $K(h)$ model by Mualem (1976)

$$K(h) = K_s\left(1 - (\alpha|h|)^{n-1}\left[1 + (\alpha|h|)^n\right]^{-m}\right)^2\left[1 + (\alpha|h|)^n\right]^{(-m/2)} \tag{5}$$

And the $\theta(h)$ model by van Genuchten (1980)

$$\theta(h) = (\theta_s - \theta_r)\left[1 + (\alpha|h|)^n\right]^{-m} + \theta_r \tag{6}$$

It is also known as the MVG model. In Eqs. (5) and (6), | | represents the absolute value, α [1/L], n [-], and m [-] are soil parameters and $m = 1-1/n$. van Genuchten and Nielsen (1985) claimed that Eq. (5) is valid over a broader range of pressure head values than the exponential model.

To take advantage of Type 1 model and Type 2 model, a third type mode (Type 3) is used in this study. In this type model, the exponential model (Eq. (2)) in Type 1 is used to describe the relationships between the hydraulic conductivity and the pressure head. For the moisture content and the pressure head relationship, Eq. (6) of Type 2 model is used. Specifically, Type 3 uses the following relationships:

$$\bar{k} = k_s e^{(\beta h)} \tag{7}$$

for hydraulic conductivity and pressure head relationship and

$$\theta(h) = (\theta_s - \theta_r)\left[1 + (\alpha|h|)^n\right]^{-m} + \theta_r \tag{8}$$

for moisture and pressure head relationship. Eqs. (7) and (8) are our Type 3 model in this study. Using this model, we derive the following function:

$$\frac{d\theta}{d\bar{k}} = \frac{\alpha m n(\theta_s - \theta_r)\left(\frac{\alpha}{\beta}\left|\log\left(\frac{\bar{k}}{k_s}\right)\right|\right)^{n-1}}{\beta\bar{k}\left[1 + \left(\frac{\alpha}{\beta}\left|\log\left(\frac{\bar{k}}{k_s}\right)\right|\right)^n\right]^{m+1}} \tag{9}$$

where α is a soil pore-size distribution parameter (1/cm), k_s is the saturated hydraulic conductivity of soil (cm/h), θ_s and θ_r are the saturated moisture content (cm^3/cm^3) and residual moisture content (cm^3/cm^3), respectively. Further, n is a function of the pore size distribution, $m = 1 - 1/n$, and β is a constant sorptivity number. Ghezzehei et al. (2007) proposed a generalized conversion formulae that relates the sorptivity to n and α (i.e.,$\beta \approx 1.3n\alpha$) of type 2 model. When $n \geqslant 1.5$, $\beta \approx \frac{\alpha}{(-1.674m^4+7.192m^3-11.4m^2+6.852m-0.970)}$, which is called the capillary length method. Generally speaking, using the generalized conversion formulae, type 3 model matches with Type 2 model closer than using the capillary length method. However, when n approaching 2, using both of formulae, type 3 model yield behaviors closely resembling that of Type 2 model.

Since Type 2 relationships are complex nonlinear functions, it is difficult to implement them into FAMM. In the following analysis, FAMM will be formulated using Type 1 and Type 3 for the unsaturated hydraulic conductivity function and moisture retention function in this study. Appendix A provides the FAMM formulation for constitutive models by Campbell (1985), and Brooks and Corey (1964).