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Analytic elements of smooth shapes

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ABSTRACT

We present a method for producing analytic elements of a smooth shape, obtained using conformal mapping. Applications are presented for a case of impermeable analytic elements as well as for head-specified ones. The mathematical operations necessary to use the elements in practical problems can be carried out before modeling of flow problems begins. A catalog of shapes, along with pre-determined coefficients could be established on the basis of the approach presented here, making applications in the field straight forward.

1. Introduction

The analytic element method is based on the superposition of analytic functions, each representing a certain feature in an aquifer. Some of these elements consist of lines that can be combined to model such features as impermeable boundaries, rivers, lake boundaries, fractures, or boundaries of inhomogeneities. More advanced analytic elements represent an entire feature, such as a lake, an impermeable boundary, or an inhomogeneity. The analytic elements all possess degrees of freedom that can be determined to meet boundary conditions, often to within machine precision.

Advanced analytic elements have been created for a variety of purposes. Line elements (Strack and Haitjema, 1981a; Strack and Haitjema, 1981b; Strack, 1989; Janković and Barnes, 1999); curvilinear elements (Strack, 1989; Steward et al., 2008), circular analytic elements, (Barnes and Janković, 1999), and analytic elements with elliptic boundaries (Strack, 1989; Suribhadla et al., 2004; Strack, 2005), have been developed. These elements have been applied to solve practical problems of groundwater flow, e.g., Bakker et al. (1999). A different application of advanced analytic elements is to create a numerical laboratory which makes it possible to study phenomena such as contaminant transport and research into equivalent hydraulic conductivity (Janković et al., 2003; Janković et al., 2006).

We may divide analytic elements applicable to the Laplace and Poisson equations in two groups; the first group consists of straight or curvilinear line-elements that can be combined to form open or closed strings of elements. The second group consists of elements bounded by closed contours, such as circles and ellipses. We present elements in this paper that are closed and can be constructed to include a variety of shapes; their construction is based on conformal mapping. We apply the elements to model impermeable boundaries as shown in Fig. 1, and boundaries with constant head, i.e., the elements that represent water bodies where either the discharge or the head is prescribed, see Fig. 7. The analytic elements can also be used to model inhomogeneities in hydraulic conductivity, following the approach outlined in Strack (1989), but this application is not included in this paper.

The approach makes use of the function that maps conformally the upper half plane onto the exterior of a slot with two legs of finite length as shown in Fig. 2. The slot is then modeled as an equipotential that captures flow from infinity. The shape of the element is chosen from the family of equipotentials that surround the slot. The angle between the legs, the ratio between the lengths of the legs, the angle between one of the legs and the x-axis, and the head along the chosen equipotential are the parameters that

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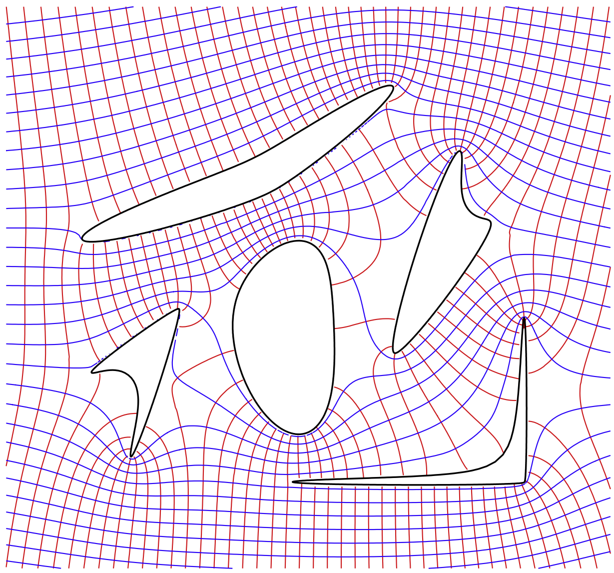

Fig. 1. Five impermeable objects in a field of uniform flow.

control the shape of the analytic element. Note that the analytic element reduces to the slot itself, when the equipotential representing the slot is chosen as the boundary of the element. Such an analytic element, or an analytic element formed by the equipotential close to the slot, may be used to model a fracture.

2. Mapping $\Im\zeta \geqslant 0$ onto the exterior of the slot

An example of a two-legged slot in the physical plane $z = x + iy$ is shown in Fig. 2. We define the dimensionless complex variable Z as

$$Z = \frac{z}{L} \tag{1}$$

where L is the length of the longest of the two legs. We number the various points of the domain in the Z-plane as 1, 2, 3, and 4. Points 1 and 3 are at the origin, but on different sides of the slot. Points 2 and 4 are the end points of the two legs. We map the upper half plane $\Im\zeta \geq 0$ onto the exterior of the slot in such a way that infinity corresponds to point 1, and $\zeta = \xi_3, \Im\xi_3 = 0$, to corner point 3. Points 2 and 4 do not enter into the transformation; only points at the origin or infinity do. Point $\zeta = \mathrm{i}$ corresponds to infinity in the z-plane. The locations of points on the real axis $\Im\zeta = 0$ that correspond to points 2 and 4 are determined later; we label the coordinates of these points as ξ_2 and ξ_4.

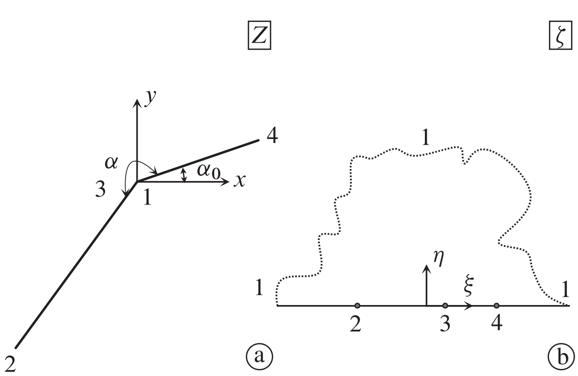

Fig. 2. The slot in the z-plane (a) and the upper half ζ-plane (b).

The function that maps the upper half plane shown in Fig. 2b onto the slot in Fig. 2a is given by the transformation of piecewise constant argument, see, e.g., Koppenfels and Stallmann (1959), Polubarinova-Kochina (1977), Strack (1989)

$$Z = |A|\mathrm{e}^{\mathrm{i}\alpha_0}\frac{(\zeta-\xi_3)^{\alpha/\pi}}{(\zeta+\mathrm{i})(\zeta-\mathrm{i})} \tag{2}$$

where $|A|$ controls the size of the element, and α_0 is given by its orientation. If only a single element were considered, the angle α_0 could be left out, but since analytic elements are designed to be superimposed, the orientation of the individual elements must be included. The constant α represents the angle between the slots as indicated in Fig. 2. The parameter ξ_3 is real; its value depends on the ratio of the lengths of the legs of the slot. The function (2) has poles of the first order at both $\zeta = \mathrm{i}$ and $\zeta = -\mathrm{i}$. Only the pole at $\zeta = \mathrm{i}$ lies in the upper half plane; it corresponds to infinity in the Z-plane. The pole $\zeta = -\mathrm{i}$ lies in the lower half plane; it does not correspond to a point in the physical plane. The function (2) contains two parameters: $|A|$ and ξ_3, which are determined from both the ratio of the lengths of the two legs and the scale. The angle α_0 is the angle between leg 3-4-1 and the x-axis, and α is the angle between legs 4-3 and 3-2, see Fig. 2. The denominator of the mapping function is real and positive for real values of ζ; it represents the square of the modulus of $\zeta - \mathrm{i}$. We show that the mapping function (2) indeed maps the upper half plane onto the Z-plane.

We have $\zeta = \xi \geqslant \xi_3$ along section 3-4-1; recall that ξ_3 is the value of $\zeta = \xi$ at point 3. The argument of the function $Z = Z(\zeta)$ is equal to α_0 along 3-4-1, which means that Z indeed represents a point on that slot. Point 1 corresponds to infinity in the ζ-plane and the mapping function indeed vanishes for $\zeta \to \infty$. The argument of $\zeta - \xi_3$ is π along 1-2-3; the argument of the mapping function is $\alpha_0 + \alpha$ along that section, which is again as desired.

The derivative of the mapping function, $Z = Z(\zeta)$, with respect to ζ is zero at the end points of the legs, points 2 and 4; the direction of the increment dZ in the derivative $dZ/d\zeta$ changes while the increment $d\zeta$ remains positive as the point ζ travels along the real axis in positive direction, which results in a change of sign. We determine the logarithmic derivative of the mapping function by differentiating the logarithm of Z with respect to ζ, which gives

$$\ln Z = \ln|A| + \mathrm{i}\alpha_0 + \sum_{m=1}^{3}\kappa_m \ln(\zeta - \zeta_m) \tag{3}$$

where

$$\zeta_1 = \mathrm{i} \qquad \zeta_2 = -\mathrm{i} \qquad \zeta_3 = \xi_3 \tag{4}$$

and

$$\kappa_1 = \kappa_2 = -1 \qquad \kappa_3 = \frac{\alpha}{\pi} \tag{5}$$

The derivative of this function with respect to ζ is

$$\frac{d\ln Z}{d\zeta} = \frac{Z'(\zeta)}{Z(\zeta)} = \sum_{m=1}^{3}\frac{\kappa_m}{\zeta-\zeta_m} = \frac{-1}{\zeta-\mathrm{i}} - \frac{1}{\zeta+\mathrm{i}} + \frac{\alpha}{\pi}\frac{1}{\zeta-\xi_3} \tag{6}$$

The function $Z(\zeta)$ is unequal to zero at points 2 and 4, where $Z'(\zeta)$ is zero; we find the zeros of the derivative of the mapping function by setting (6) equal to zero

$$\frac{\alpha}{\pi}(\zeta^2+1) - (\zeta-\xi_3)[\zeta+\mathrm{i}+\zeta-\mathrm{i}] = 0 \tag{7}$$

The two roots of this quadratic equation are

$$\zeta_{1,2} = \mu_{1,2} = \frac{\xi_3}{2-\alpha/\pi} \pm \sqrt{\left[\frac{\xi_3}{2-\alpha/\pi}\right]^2 + \frac{\alpha/\pi}{2-\alpha/\pi}} \tag{8}$$