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An analytical model for steady vertical flux through unsaturated soils with special hydraulic properties

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SUMMARY

An analytical solution for one-dimensional steady vertical flux through unsaturated homogeneous soils is presented. The model assumes power law hydraulic conductivity and diffusivity functions. The soil domain is a finite-depth flow medium overlying a water table. A steady constant flux is applied at the top boundary while a constant saturation value is specified at the bottom boundary. The general form of the analytical solution expresses implicitly the depth as function of the liquid water saturation. It can be used to model both infiltration through the soil surface and evaporation from the bottom, depending on the sign of the flux boundary value. The analytical solution takes into account the prediction of a drying front in the case of evaporation from deep water table. Algebraic expressions of practical and theoretical importance are derived in terms of soil water parameters. These expressions include the stored mass in the system at steady state as well as the drying front when it exists. The general form solution can be inverted back to obtain exact explicit solutions when the power law parameters are related. Numerical results show the effects of soil type, surface flux, capillarity, and gravity on the saturation distribution in the soil. The analytical solution is used for comparing between models, validating of numerical solutions, as well as for estimating the hydraulic parameters.

1. Introduction

Analysis of water flow through the vadose zone is important in a broad range of engineering disciplines, including agriculture and irrigation, hydrologic budget, aquifer recharge, climate change studies and waste management. The determination of many factors such as the distribution, location, quantity and residence time of water in the unsaturated zone depends on the coupling between the atmospheric and subsurface branches of the hydrologic cycle. The moisture state in the unsaturated zone is strongly related to the season and the changes in climate during storms and interstorm periods (Salvucci, 1993).

During wet seasons water is infiltrated from the soil surface due to precipitation and may reach rapidly the steady state in the region located between the soil surface and the water table (WT). However, in dry seasons the flow of water above the WT is mainly governed by evaporation and plant transpiration. In such seasons the evaporation process may be considered using two general approaches (Gowing et al., 2006; Lehmann et al., 2008; Shokri and Salvucci, 2011; Sadeghi et al., 2012). The first approach applies when the WT level is shallow enough so that the hydraulic connections between this later and the soil surface are maintained. The second approach corresponds to relatively deep WT levels indicating hydraulic disconnections. In that case the evaporation plane evolves from an intermediate depth located between the WT and the soil surface and is referred to as the drying front (DF) (Sadeghi et al., 2012).

Analyses of steady state vertical flux through unsaturated soils may well be addressed using simplified one-dimensional analytical models. These analytical models are useful for providing initial conditions for flow models, conducting sensitivity analyses to investigate the effects of various parameters on water flow in the vadose zone, serving as screening models, estimating hydraulic parameters from laboratory or field experiments, and providing benchmark solutions for more complex flow problems that cannot be solved analytically. Various analytical solutions of one-dimensional steady state vertical flux through homogeneous soils have been reported in the literature. Among these works we cite those of Gardner (1958), Warrick (1988), Salvucci (1993), Basha (1999), Zhu and Mohanty (2002), and Sadeghi et al. (2012). In all these cited works the hydraulic parameters are based on specific models such as the exponential model, the rational power model (Gardner, 1958) and the Brooks–Corey model (Brooks and Corey, 1964).

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In this paper we present a general analytical model for the one-dimensional steady vertical flux problem. The model assumes power law relative permeability and diffusivity functions. Such power law functions covers well known models such as the linear model (Warrick, 2003), the models based on the Brooks and Corey soil water retention curve, the van Genuchten model under low saturation values (van Genuchten, 1980), models with constant diffusivity function, and others. The main feature of the proposed general form is that it is valid for any power exponent, so that already existing solutions based on specific types of hydraulic functions can be shown to be particular solutions of the presented analytical solution. We developed an analytical solution which is valid for both infiltration and evaporation problems. The general form of the analytical solution expresses implicitly the depth as function of the effective liquid saturation. Some exact explicit solutions are obtained from the general form when the power law parameters are related. An analytical expression of the quantity of liquid water in the system in term of liquid mass fraction is also derived. This expression may give some insight about the effect of soil hydraulic functions (i.e. relative permeability and capillary pressure) on the quantity of water stored in the system at steady state. In the case of evaporation, we provide also the analytical expression of the drying front depth when it exists. Possible uses for the proposed analytical solutions include estimating effective parameters of hydraulic properties by inverse modeling based on the analytical expressions, establishing initial conditions for numerical flow and transport models, and validating of numerical solutions.

2. Mathematical model

The one-dimensional steady-state equation describing vertical flux in unsaturated soils is obtained from the combination of the mass conservation equation and the generalized Darcy law. It can be expressed in the following form

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[-\frac{kk_r}{\mu}\left(\frac{\mathrm{d}P_l}{\mathrm{d}z}-\rho g\right)\right]=0, \tag{1}$$

where z is the soil depth (L) directed positive downward, k is the intrinsic permeability (L^2) of the porous medium, k_r, μ, ρ and P_l are the relative permeability (–), the viscosity (M/LT), the density (M/L^3) and the pressure (M/LT^2) of the liquid phase, and g is the gravity acceleration constant (9.81 m^2/s).

The liquid pressure P_l can be written as function of the capillary pressure P_c and the gas phase pressure P_g which represents air. The capillary pressure is defined as the difference between the gas and liquid pressures

$$P_c = P_g - P_l. \tag{2}$$

For unsaturated flow, the gas pressure is assumed to be constant. Substituting (2) into (1) and integrating once we get

$$\frac{kk_r}{\mu}\left(\frac{\mathrm{d}P_c}{\mathrm{d}z}+\rho g\right)=q, \tag{3}$$

where q is a constant of integration representing the infiltration/evaporation rate (L/T) applied at the soil surface (i.e., at $z = 0$). For infiltration q is positive while it is negative for evaporation.

The capillary pressure and the relative permeability functions are usually defined as functions of the effective liquid saturation S defined by

$$S=\frac{S_l-S_{lr}}{1-S_{lr}}, \tag{4}$$

in which S_l is the liquid saturation and S_{lr} is the residual liquid saturation. Note that the liquid saturation S_l ranges between S_{lr} and 1 while the effective saturation S ranges between 0 and 1.

Eq. (3) can be rewritten as

$$\frac{kk_r(S)}{\mu}\left(P_c'(S)\frac{\mathrm{d}S}{\mathrm{d}z}+\rho g\right)=q, \tag{5}$$

where P_c' is the derivative of P_c with respect to S. Note that the term $D(S)=-kk_r(S)P_c'(S)/\mu$ is called the diffusivity function.

In order to solve Eq. (5), expressions of $k_r(S)$ and $P_c'(S)$ have to be defined. In this work we use power law functions of the forms

$$k_r(S)=C_kS^{\alpha}, \tag{6}$$

and

$$P_c'(S)=-C_pS^{-\beta}, \tag{7}$$

where C_k is a positive dimensionless constant, C_p is a positive constant which have dimension of pressure, and α and β are two positive dimensionless exponents not necessary integers.

According to (6) and (7) the diffusivity function can be written as

$$D(S)=\frac{kC_kC_p}{\mu}S^{\alpha-\beta}. \tag{8}$$

The diffusivity function of unsaturated soils is an increasing function of S. This requires that $\alpha-\beta \geqslant 0$.

Hydraulic properties of types (6)–(8) are frequently used in the literature (Brooks and Corey, 1964; Honarpour et al., 1986; Wu and Pan, 2003, 2005; Hayek, 2014). The most popular capillary pressure model which has derivative of the form (7) is the Brooks–Corey (BC) capillary pressure model (Brooks and Corey, 1964) for which $P_c(S)=-P_eS^{-1/\lambda}$, where P_e is the entry pressure and λ is the pore size distribution index. For this model we have $C_p=P_e/\lambda$ and $\beta=(\lambda+1)/\lambda$. The relative permeability function can be calculated based on the expression of the capillary pressure. Several models have been proposed in the past to express the relative permeability as function of the capillary pressure (Purcell, 1949; Burdine, 1953; Mualem, 1976). Mualem and Dagan (1978) summarized these models to develop a general model based on the equation $k_r(S)=S^l\left[\int_0^S \mathrm{d}x/(P_c(x))^{\eta}/\int_0^1 \mathrm{d}x/(P_c(x))^{\eta}\right]^{\chi}$, where l and η are dimensionless parameters related to the tortuosity of the soil pores, and χ is a dimensionless parameter which can be determined by the method of evaluating the effective pore radius (Raats, 1992). For the BC-model, the general expression of the relative permeability writes $k_r(S)=S^{l+\chi+\chi\eta/\lambda}$ which is of the form (6) with $C_k = 1$ and $\alpha=l+\chi+\chi\eta/\lambda$. The van Genuchten (vG) capillary pressure model (van Genuchten, 1980) may lead also to power law functions of the form (6)–(8) under some condition, see van Genuchten and Nielsen (1985) for more details about the restriction of the vG-model to power laws of the form (6)–(8). The linear model (Warrick, 2003) can be also written in the forms (6) and (7) with $C_k = 1$ and $\alpha = \beta = 1$ (i.e. for this model the capillary pressure is a logarithmic function defined by $P_c(S) = C_p \ln(S)$). Wu and Pan (2003, 2005) proposed an analytical solution for the linearized Richards' equation in the three-dimensional space without gravity effects by assuming $\alpha = \beta$ (i.e., constant diffusivity). Hayek (2014) proposed similarity solutions for the one-dimensional Richards' equation describing the migration of a finite mass of water through a semi-infinite vertical unsaturated porous column for the particular cases ($\alpha = \beta = 2$) and ($\alpha = 3$, $\beta = 2$). In this work, we consider the general case where exponents α and β are arbitrary positive parameters verifying $\alpha-\beta \geqslant 0$.

The following dimensionless variables are introduced

$$Z=\frac{z}{H},\ Q=\frac{q}{q_0},\ \gamma=\frac{C_p}{H\rho g}, \tag{9}$$

where H is the thickness of the soil layer (L) and $q_0 = k\rho gC_k/\mu$ is the saturated hydraulic conductivity (L/T). In the above formulas Q is the dimensionless flux applied at the soil surface (the relative