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High resolution numerical schemes for solving kinematic wave equation

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SUMMARY

This paper compares the stability, accuracy, and computational cost of several numerical methods for solving the kinematic wave equation. The numerical methods include the second-order MacCormack finite difference scheme, the MacCormack scheme with a dissipative interface, the second-order MUSCL finite volume scheme, and the fifth-order WENO finite volume scheme. These numerical schemes are tested against several synthetic cases and an overland flow experiment, which include shock wave, rarefaction wave, wave steepening, uniform/non-uniform rainfall generated overland flows, and flow over a channel of varying bed slope. The results show that the MacCormack scheme is not a Total Variation Diminishing (TVD) scheme because oscillatory solutions occurred at the presence of shock wave, rarefaction wave, and overland flow over rapidly varying bed slopes. The MacCormack scheme with a dissipative interface is free of oscillation but with considerable diffusions. The Godunov-type schemes are accurate and stable when dealing with discontinuous waves. Furthermore the Godunov-type schemes, like MUSCL and WENO scheme, are needed for simulating surface flow from spatially non-uniformly distributed rainfalls over irregular terrains using moderate computing resources on current personal computers.

1. Introduction

The kinematic wave equation was first developed by Lighthill and Whitham (1955). The equation is based on the assumptions that the acceleration term and the pressure gradient term in the momentum equation are negligible, so that the energy slope is equal to the bottom slope. The kinematic wave model is commonly used to simulate the overland flow (Ponce, 1991; Singh, 2001). Henderson (1966) showed that natural flood waves behave nearly the same as the kinematic wave in steep slopes ($S_0 > 0.002$). Vieira (1983) concluded that the kinematic wave equation can be used on natural slopes with the kinematic wave number, $k \gg 50$. Ponce (1991) compared the kinematic wave equation with the unit hydrograph as a practical method of overland flow routing. Singh (2001) concluded that the kinematic wave equation is applicable to surface water routing, vadose zone hydrology, riverine and costal processes, erosion and sediment transport, etc.

The kinematic wave equation is a first-order hyperbolic partial differential equation (PDE). For a hyperbolic equation, the disturbance will travel along the characteristics of the equation in a finite propagation speed. This feature distinguishes the hyperbolic equations from elliptic and parabolic equations. On the other hand, the kinematic wave equation also belongs to a kind of equations called conservation laws (LeVeque, 2002; Toro, 2009). Since the flux term is a nonlinear function of conservative variables, the solution does not propagate uniformly but deforms as it evolves. Even the initial conditions are continuous and smooth, the hyperbolic conservation laws can develop discontinuities in the solution, for example, shock waves.

Both shock and rarefaction waves are the intrinsic features of hyperbolic equations. Lighthill and Whitham (1955) discussed the formations of shock wave and rarefaction wave. Kibler and Woolhiser (1970) investigated the structure and general properties of shock waves and developed a numerical procedure for shock fitting. Eagleson (1970) found that using non-uniform flow depth as initial condition, non-uniform rainfall in the source term, or increasing inflows as the boundary condition may cause the formation of kinematic shock wave. Borah et al. (1980) presented the propagating shock-fitting scheme (PSF) to simulate overland flow with shock waves. Singh (2001) found three factors that affect the shock wave formation: (1) initial and boundary conditions; (2) lateral inflow and outflow, and (3) watershed geometric characteristics. Due to the complex geometry, non-uniform roughness

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Nomenclature

c	wave celerity (m/s)
f	infiltration rate (mm/h)
h	flow depth (m)
h_i	flow depth at the center of cell i (m)
$h_{i+1/2}$	reconstructed depth at the interface $i + 1/2$ of cell (m)
h_L, h_R	flow depth at the left and right of wave front (m)
i_0	rainfall excess (mm/h)
i	intensity of rainfall (mm/h)
$L(h)$	operator $L(h)$ defined in Eq. (12) (–)
L	channel length (m)
m	exponential in Eq. (19) (–)
n	Manning's roughness coefficient ($sm^{-1/3}$)
q	discharge per unit width (m^2/s)
q_L	outflow discharge (m^2/s)
$r_{i+1/2}$	distance from the cell center to the interface $i + 1/2$ (m)
S	shock wave speed (m/s)
S_0	bed slope (–)
$TV(h)$	total variation of flow depth (–)
t	simulation time (s)
t_r	duration of rainfall (s)
t_c	time of concentration (s)
t_p	time defined in Eq. (31) (s)
v	flow velocity (m/s)
x	downslope distance (m)

Subscript

i	spatial index

Superscript

n, $\overline{n+1}$, and $n + 1$ and $n + 1,^*$	temporal indices

Greek

α	coefficient in Eq. (19) (–)
γ_j	linear weights (–)
θ	coefficient for the dissipative interface (–)
$\varepsilon = 10^{-6}$	truncation error for actual calculations (–)
Δt	time step (s)
Δx	space step (m).
∇h_i	limited depth gradient of cell i (–)
∇h_L, ∇h_R	limited depth gradient at the left and right of cell interface (–)

and non-uniform rainfall pattern, it is impossible to derive a general analytical solution for the kinematic wave equation. Singh (2001) summarized three numerical techniques for solving the kinematic wave equation: (1) method of characteristic, (2) Lax–Wendroff finite difference method, and (3) finite element method. Numerical diffusion and numerical dispersion were observed when using the finite difference schemes (Ponce, 1991). Kazezyılmaz-Alhan et al. (2005) evaluated several finite difference schemes for solving kinematic wave equation: the linear explicit scheme, the four-point Pressimann implicit scheme, and the MacCormack scheme. The study (Kazezyılmaz-Alhan et al., 2005) found the MacCormack scheme is better than the four-point implicit finite difference scheme for shock capture. However, Kazezyılmaz-Alhan et al. (2005) did not explicitly examine the dispersion occurred at the shock and rarefaction waves from non-uniformly distributed rainfall. The stability of the classical MacCormack scheme at the presence of shock and rarefaction wave remains unknown.

Recently, the Godunov-type finite volume method has been widely used in solving shallow water equations (LeVeque, 2002; Toro, 2009) because of its wide applicability, strong stability, and high accuracy. One of the most popular Godunov-type methods is a second-order, TVD (Total Variation Diminishing) scheme, namely the MUSCL (Monotone Upstream-centered Schemes for Conservation Laws) scheme (van Leer, 1979). The MUSCL scheme is a high-resolution scheme because (1) the spatial accuracy of the scheme is equal to or higher than second order; (2) the scheme is free from numerical oscillations or wiggles; (3) high-resolution is produced around discontinuities. In general, the high-resolution schemes are considered as tradeoffs between computational cost and desired accuracy (Harten, 1983; Toro, 2009). Another popular but relatively new method is the high-order WENO (Weighted Essentially Non-Oscillatory) finite volume scheme (Shu, 1999). High-order means the order of accuracy is equal to or higher than the third-order. According to Shu (2009), the WENO scheme is suitable for the complicated problems, such as flow having both shocks and complicated smooth structures (e.g., small perturbation). Although the computational cost of high-order WENO scheme can be three to ten times than a second-order high-resolution scheme, it is still preferable because of its high-order accuracy in both time and space. The applications of those two high resolution schemes to solve the kinematic wave equation have not been studied. Whether or not these finite volume schemes have advantages over the commonly used finite difference schemes are examined in this paper.

This study compares the Godunov-type finite volume method using MUSCL scheme and WENO scheme with the finite difference method using MacCormack scheme. The paper is organized as follows: Section 2 introduces the kinematic wave equation and its analytical solutions; Section 3 discusses the numerical schemes: the MacCormack scheme, the MUSCL scheme and the WENO scheme; Section 4 shows the results of typical test cases. Finally, several concluding remarks are given in Section 5.

2. Governing equations

The one-dimensional kinematic wave equation for flows over a slope is given by (Eagleson, 1970; Lighthill and Whitham, 1955):

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = i_0 \tag{1}$$

where h is the depth of flow; q is the discharge per unit width; $i_0 = i - f$ is the rain excess; i is the intensity of rainfall; f is the infiltration rate; t is the time; x is the downslope distance.

For the overland flow, the discharge q is defined as:

$$q = \alpha h^m \tag{2}$$

where m is the exponential, and α is the coefficient. For fully turbulent flow, the coefficients are given by Ponce (1989):

$$\alpha = \frac{1}{n}\sqrt{S_0}, \quad m = \frac{5}{3} \tag{3}$$

where n is the Manning's roughness coefficient; S_0 is the bottom slope. It is obvious that the flux function $q(h)$ is a convex function (Jacovkis and Tabak, 1996; Toro, 2009) because the second order derivative is positive:

$$\frac{d^2q}{dh^2} = \alpha m(m-1)h^{m-2} > 0, \quad \text{for } h > 0 \tag{4}$$

The analytical solution for Eq. (1) has been found by (Eagleson, 1970) in which the outflow hydrograph is a function of rainfall intensity and the time of concentration.