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Water pulse migration through semi-infinite vertical unsaturated porous column with special relative-permeability functions: Exact solutions

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SUMMARY

The paper presents certain exact solutions describing the vertical movement of a water pulse through a semi-infinite unsaturated porous column. The saturation-based form of the Richards' equation is used with special power law relative-permeability functions. Both capillary and gravity effects are taken into account. Three exact solutions are derived corresponding to three relative-permeability functions, linear, quadratic and cubic. The Richards' equation is nonlinear for the three cases. The solutions are obtained by applying a general similarity transformation. They are explicit in space and time variables and do not contain any approximation. They describe the evolution of the water saturation in the vertical column and they can be used to predict the post-infiltration movement of a finite quantity of water. Exact expressions of the masses of water leaving a given depth are also derived for the three cases. We analyze the effect of relative-permeability and capillary pressure. The proposed solutions are also useful for checking numerical schemes. One of the exact solutions is used to validate numerical solution obtained from an arbitrary initial condition. Results show that the numerical solution converges to the exact solution for large times.

1. Introduction

The unsaturated zone, also called the vadose zone, plays a crucial role in subsurface hydrology and irrigation engineering because this zone controls, to a large degree, the transmission of water to aquifers and it is often regarded as a filter which removes undesirable substances before they affect aquifers (Stephens, 1996; Selker et al., 1999). In the past few decades, problems related to infiltration through the unsaturated zone have received more attention than those related to the post-infiltration stage by soil physicists and hydrologists. However, prediction and understanding of the movement of water that has infiltrated into the unsaturated zone is an important problem in hydrology and irrigation engineering since it determines the amount of water stored near the soil surface and the range of time that this quantity of water remains available for plant uptake (Sander et al., 1991). Consequently, a considerable amount of effort has been devoted in soil science to the mathematical modeling of water redistribution in the unsaturated zone (Warrick, 2003; Philip, 1991; Sander et al., 1991; Raats and van Duijn, 1995; Wallach and Jortzick, 2008; Pop et al., 2009; Doster et al., 2012).

The vertical movement of water in the unsaturated zone is generally governed by two forces: capillarity and gravity. Richards' equation is traditionally used to describe the water movement in the unsaturated zone. This equation can be written in different forms: water content-based form, pressure-based (hydraulic head) form, mixed form or the saturation-based form. Wu and Pan (2003) used the saturation-based form by neglecting gravity effects to study the transient flow into unsaturated rock matrix. They linearized the Richards' equation by using special relative-permeability and capillary pressure functions. In this paper we adopt this form of the Richards' equation and we additionally take into account gravity effects. This form is a simplified representation of a two-phase flow model where the two phases are water and air, respectively. The main factors which allow such simplification are: (1) the low density of air which is about three orders of magnitude less than the density of water, leading to negligible changes in pressure in the vertical direction; (2) the low viscosity of air which is about two orders of magnitude less than the viscosity of water, which means that air can move under very small air pressure gradient. Accordingly, the Richards' equation can be written as a nonlinear advection–diffusion equation where the primary variable is the water saturation. Nonlinearity comes from the complicated relationships between relative-permeability—saturation and capillary pressure—saturation. Because of the nonlinear nature of the

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Richards' equation, exact analytical solutions are generally difficult to obtain. Several papers dealing with analytical solutions of the nonlinear Richards' equation have been published over the past few decades. Among these papers we cite the works of Philip (1969), Sander et al. (1988), Broadbridge and White (1988), Zimmerman and Bodvarsson (1989, 1990, 1995), Warrick et al. (1990, 1991), Kühnel et al. (1990), Sander et al. (1991), Marinelli and Durnford (1998), Serrano (2004), Triadis and Broadbridge (2010), Basha (2011), and Nasseri et al. (2012). Analytical solutions exist also for the full system of two-phase flow which may be used also as solutions of the Richards' equation after making the above mentioned simplification (see for instance, McWhorter and Sunada (1990), van Duijn and De Neef (1998), and Fučík et al. (2007, 2008, 2010)). Some of these analytical solutions are approximate solutions and the others are relatively complex to use since they are presented in parametric form or infinite series. Exact explicit solutions, if they exist, provide good insight into many subsurface flow problems and they can be considered as reference solutions for checking the accuracy of various numerical methods.

Based on the saturation form, we propose some exact solutions of the one-dimensional Richards' equation by using special power law relative-permeability and capillary pressure functions. Such power law functions are frequently used in the literature (Brooks and Corey, 1964; Campbell, 1985; Honarpour et al., 1986; Warrick, 2003; Wu and Pan, 2003, 2005). We derive three exact solutions for nonlinear and non-hysteretic migration of a finite water pulse through a semi-infinite column taking into account capillary and gravity effects. The three exact solutions correspond to three relative-permeability functions: linear, quadratic and cubic. Several authors used these relative-permeability functions in the literature (see, e.g., (Garg et al., 1996; Warrick, 2003; Tracy, 2008) for linear relative-permeability, (Forsyth, 1987; Efendiev and Hou, 2009) for quadratic relative-permeability and (Udell and Fitch, 1985; Lu et al., 2009) for cubic relative-permeability). This later (i.e. cubic) corresponds to $\lambda = 1$ in the Purcell model (Purcell, 1949; Li and Horne, 2006). Considering such special relative-permeability functions may be seen as a limitation. However, we believe that such special functions can be matched well enough to provide useful engineering predictions. The exact solutions are obtained by using a general similarity transformation. They describe the evolution in space and time of the water saturation in a vertical column and they do not require any numerical implementation. The solutions for horizontal flow (i.e. without gravity) are also presented. Analytical expressions of the masses of water leaving a given depth are derived for the three cases. The analytical expressions of masses may give some insight about the effect of hydraulic soil functions (i.e. relative-permeability and capillary pressure) on the length of time that water remains available in the system. One of the proposed exact solutions is used to compare with numerical solution obtained from an arbitrary initial condition.

2. The mathematical model

The movement of the two incompressible fluids, water and air, in a vertical porous column is governed by the phase equations of the two fluids. The phase equation of each fluid is obtained by combining the mass balance equation and the extended Darcy equation for multiphase system. Under the assumption of low air density and low air viscosity the movement of water is described by the Richards' equation

$$\phi \frac{\partial S_w}{\partial t} = \frac{\partial}{\partial z}\left(\frac{kk_{rw}}{\mu_w}\left(\frac{\partial P_w}{\partial z} - \rho_w g\right)\right), \tag{1}$$

where t is the time (T), z is the vertical coordinate directed downward (L), ϕ and k are the porosity $(-)$ and the intrinsic permeability (L^2) of the porous material, k_{rw}, μ_w, P_w and S_w are the relative-permeability $(-)$, the viscosity (M/LT), the pressure (M/LT^2) and the saturation of water $(-)$.

The relative-permeability k_{rw} and the capillary pressure P_c which is defined as the difference between the air pressure P_a and the water pressure P_w, are functions of the water saturation S_w. In this paper we select power law functions as in (Honarpour et al., 1986; Wu and Pan, 2003, 2005) in the form

$$k_{rw}(S_w) = C_k (S_w^*)^{\alpha}, \tag{2}$$

and

$$P_c(S_w) \equiv P_a - P_w = C_p (S_w^*)^{-\beta}, \tag{3}$$

where C_k and C_p are constant coefficients, α and β are positive constant exponents of the relative-permeability and the capillary pressure functions, respectively, and S_w^* is the effective water saturation defined by

$$S_w^* = \frac{S_w - S_{wr}}{1 - S_{wr}}, \tag{4}$$

with S_{wr} is the residual water saturation. We note that S_w ranges from S_{wr} to 1 while S_w^* ranges from 0 to 1.

Assuming that air is at constant pressure, say at atmospheric pressure, therefore the gradient of the water pressure is the opposite of the gradient of the capillary pressure. Substituting (2)–(4) into (1) we get after some mathematical manipulations

$$\frac{\partial S_w^*}{\partial t} = D_0 \frac{\partial}{\partial z}\left((S_w^*)^{\alpha-\beta-1}\frac{\partial S_w^*}{\partial z}\right) - V_0 \frac{\partial}{\partial z}\left((S_w^*)^{\alpha}\right), \tag{5}$$

with D_0 has dimension of diffusion coefficient (L^2/T) and V_0 has dimension of velocity (L/T) defined by

$$D_0 = \frac{\beta k C_k C_p}{\phi \mu_w (1 - S_{wr})} \quad \text{and} \quad V_0 = \frac{\rho_w g k C_k}{\phi \mu_w (1 - S_{wr})}. \tag{6}$$

Eq. (5) is a nonlinear advection–diffusion equation. Diffusion is due to capillary forces represented by D_0 and advection is due to gravity forces represented by V_0. Note that both advection and diffusion terms are generally nonlinear.

The diffusivity $D(S_w^*) = D_0 (S_w^*)^{\alpha-\beta-1}$ is a function of saturation. For $\alpha - \beta - 1 < 0$, it decreases with water saturation. Therefore, the diffusivity tends to infinity for vanishing saturation. This means that the saturation profile is smoothly vanishing at infinity. For the other case $\alpha - \beta - 1 > 0$, the diffusivity increases with saturation and becomes negligible for small saturation values. Contrary to the previous case, the saturation profile vanishes at finite depth. Thus, the saturation profile is characterized by a shock-type moving front. We would like to mention that the first case (i.e. $\alpha - \beta - 1 < 0$) appears to be unrealistic from a physical standpoint when modeling flow in the unsaturated zone. However, such situation may exist in other systems of two-phase flow like gas and oil. For these systems, this case corresponds to circumstances where the characteristic pore size of the porous medium is small and/or the viscosity of oil is low. Moreover, this case may be important for validating numerical solutions.

Wu and Pan (2003) solved analytically Eq. (5) without gravity effects (i.e. $V_0 = 0$) for the special case $\alpha = \beta + 1$ in one, two and three dimensions. In this case the equation is reduced to a linear diffusion equation. Generally, when gravity forces are considered, Eq. (5) is not linear anymore. Indeed, Eq. (5) is linear if and only if $\alpha = \beta + 1$ and $\alpha = 1$ which yields $\beta = 0$ and then the capillary pressure is constant. In this paper we disregard this case and we assume that $\beta = 1$. Wu and Pan (2003) used this value of β in their numerical applications.

In order to find analytical solution of (5), we consider a semi-infinite porous column and we assume that a fixed amount of