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Steady-state groundwater recharge in trapezoidal-shaped aquifers: A semi-analytical approach based on variational calculus

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SUMMARY

This study presents a semi-analytical solution for steady groundwater flow in trapezoidal-shaped aquifers in response to an areal diffusive recharge. The aquifer is homogeneous, anisotropic and interacts with four surrounding streams of constant-head. Flow field in this laterally bounded aquifer-system is efficiently constructed by means of variational calculus. This is accomplished by minimizing a properly defined penalty function for the associated boundary value problem. Simple yet demonstrative scenarios are defined to investigate anisotropy effects on the water table variation. Qualitative examination of the resulting equipotential contour maps and velocity vector field illustrates the validity of the method, especially in the vicinity of boundary lines. Extension to the case of triangular-shaped aquifer with or without an impervious boundary line is also demonstrated through a hypothetical example problem. The present solution benefits from an extremely simple mathematical expression and exhibits strictly close agreement with the numerical results obtained from Modflow. Overall, the solution may be used to conduct sensitivity analysis on various hydrogeological parameters that affect water table variation in aquifers defined in trapezoidal or triangular-shaped domains.

1. Introduction

As an important component of never-ending hydrologic cycle, the infiltrating precipitation eventually rehabilitates aquifer systems through the diffusive recharge. This often occurs over large spatial scales with the recharge rates controlled by weather, soil properties, and land use (Healy, 2010). The recharge can be considered as an external forcing for the waterbudget models.

Given their advantages, several mathematical models have been developed for groundwater recharge problems. These may offer preliminary insights regarding engineering and management decisions. In particular, groundwater models relying on analytical solutions appear to be very useful, thanks to their ease of implementation while providing valuable insights. Besides, the analytical approach is well suited in verifying more comprehensive numerical schemes.

Transient-state groundwater flow can be well described by Boussinesq equation under Dupuit's assumption of hydrostatic pressure distribution. However, the strong nonlinearity of the equation often precludes obtaining an analytical solution, except for limited number of cases (Polubarinova-Kochina, 1962; Serrano, 1995). Therefore, it is common practice to linearize the Boussinesq equation in order to attain an acceptable solution for the problem (Rai et al., 2006; Liang and Zhang, 2012a). On this basis, the maximum rise or decline of the water table should remain small compared to the initial water table height (Teloglou and Bansal, 2012).

There are various analytical studies dealing with groundwater recharge in the framework of linearized Boussinesq equation. These include, but not limited to, localized recharge from overlying basins (Rai and Manglik, 1999; Manglik et al., 2004) and diffusive recharge in 1D configuration (Anderson and Evans, 2007; Liang and Zhang, 2012b). Groundwater models combining recharge basins with multiple injection and/or extraction wells can also be found, for example in the work of Chang and Yeh (2007) and Rai and Manglik (2012). Conventionally, the solution strategy relies on Laplace transform for 1D (Hernandez and Uddameri, 2013) and Fourier sine/cosine transform for 2D applications (Rai et al., 2006; Manglik et al., 2013). Also, transient flow field in the wedge-shaped aquifers can be modeled by successively applying the finite sine and Hankel transforms (Yeh and Chang, 2006; Yeh et al., 2008).

Steady-state groundwater flow condition may be regarded as asymptotic convergence of the transient response in the limit when time approaches infinity. Introducing the concept of mean

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action time, Simpson et al. (2013) evaluated a finite estimate of the time span required for a transient flow problem to attain efficiently the steady state. They considered recharge and discharge processes and confirmed their theoretical predictions with relevant laboratory-scale experiments. Under steady-state condition, Boussinesq equation loses the associated time-dependent character, simplifying to Poisson equation for 2D groundwater movement in homogeneous isotropic aquifers. The latter requires much simpler mathematical treatment compared to that of transient groundwater flow models. Some mathematically simple but useful solutions of Poisson equation are reported by Fitts (2013).

In common hydrogeological settings, the aquifer may be bounded laterally by streams or impermeable formations along one or more of its sides. Most of the analytical studies reported in the literature, however, assume semi-infinite, rectangular or wedge-shaped domains to represent laterally bounded aquifers. Difficulties arise when configuration of aquifer boundaries leads to geometries different from those mentioned above. This is the case in various regions of the world. The Mekong and Red River deltas are typical examples of triangular-shaped aquifers as reported by Asadi-Aghbolaghi and Seyyedian (2010) and references therein. Different aquifer geometries could be characterized through the intersection of streams in a multiple river basin. As an example, the intersection of Karun and Bahmanshir rivers in Khuzestan plain delineates a region with roughly trapezoidal boundary (enclosed by dashed line in Fig. 1).

Apart from those devoted to wedge-shaped aquifers, there are few analytical studies addressing flow field in non-rectangular aquifers (Asadi-Aghbolaghi and Seyyedian, 2010; Mahdavi and Seyyedian, 2013). This may be attributed to inadequacy of conventional solution techniques in dealing with complex aquifer geometries.

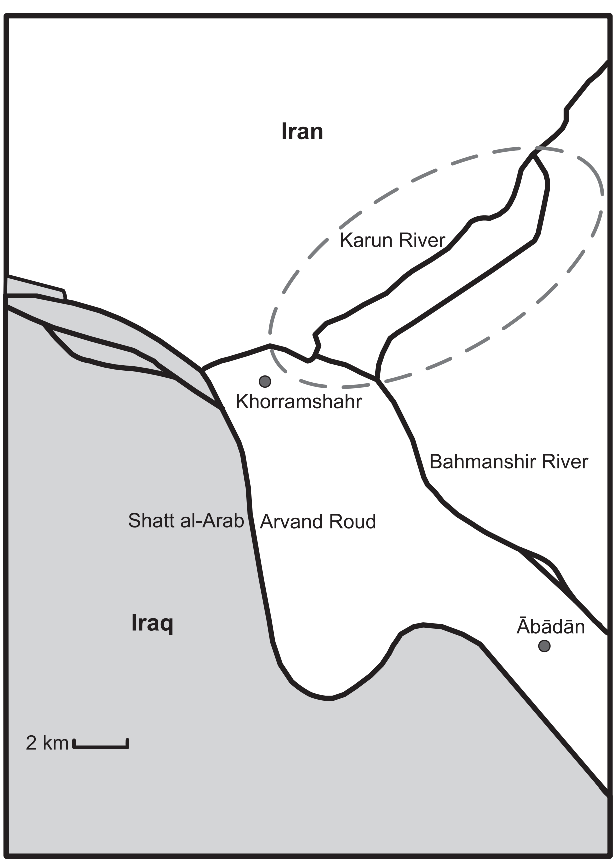

Fig. 1. Map of region bounded by Karun and Bahmanshir rivers in the vicinity of Iran–Iraq border. The dashed line depicts roughly a trapezoidal-shaped aquifer.

The present study is aimed at obtaining a semi-analytical solution for steady-state groundwater recharge in trapezoidal-shaped aquifers — an idealization of what indicated in Fig. 1. The aquifer is assumed to be homogeneous but anisotropic and it is surrounded by four constant-head streams. To the best of author's knowledge, no analytical or even semi-analytical solution exists for the problem under consideration. In accordance with the well known Kantorovich method, an admissible solution containing two unknown functions is considered. A variational framework is then constructed whereby the two functions are determined by minimizing an appropriate penalty function. This leads to a pair of coupled Euler–Lagrange ordinary differential equations (ODEs). The concept of operational calculus (Smirnov, 1964) is effectively utilized to uncouple the equations. Finally, hypothetical examples demonstrating the validity of the solution technique are presented. The model predictions are in strictly close agreement with those of Modflow simulation for both isotropic and anisotropic test cases.

2. Mathematical model

2.1. Governing equation and associated boundary conditions

This study focuses on mathematical description of steady groundwater flow in trapezoidal-shaped aquifers subject to areal diffusive recharge. Few simplifying assumptions are made in order to arrive at a reasonable mathematical model for the phenomenon. These are (i) the groundwater flows horizontally in a fully saturated porous media; (ii) the unconfined aquifer is homogeneous but anisotropic with respect to horizontal hydraulic conductivity; (iii) the aquifer is underlain by a horizontal impervious bed and receives an areal recharge of constant rate; (iv) the streams adjacent to aquifer are considered to be constant-head, fully penetrating and in prefect hydraulic connection with the aquifer; and (v) the coordinate axes are oriented along the principal directions of anisotropy. Under these circumstances, the spatial distribution of water table height is governed by:

$$\frac{\partial^2 H}{\partial x^2} + \psi \frac{\partial^2 H}{\partial y^2} = -\frac{2R_0}{K_x} \tag{1}$$

where $H = h^2 - h_0^2$ [L^2], h [L] is the water table height at the position (x, y), R_0 [L/T] is the constant rate of recharge, $\psi = K_y/K_x$ [−] is the anisotropy factor and K_x, K_y [L/T] are the hydraulic conductivities in x- and y-direction, respectively, which are considered to be constant. It is assumed that the stream–aquifer system preserves an initial equilibrium state at height h_0 [L], which is equal to the water depth in the surrounding streams. Obviously, Eq. (1) reduces to Poisson equation for the case of isotropic aquifer.

The geometry of aquifer system is schematically depicted in Fig. 2. The parameters a [L], b [L], m_1 [−] and m_2 [−] delineate the aquifer domain Ω in the Cartesian coordinate system as $\Omega = \{(x,y)|(x,y) \in (a \leqslant x \leqslant b; -m_1 x \leqslant y \leqslant m_2 x)\}$. The constant-head streams flowing adjacent to the aquifer serve as boundary of the flow domain. Mathematically speaking, these streams are included by specifying zero boundary values for the parameter H along desired border. This reflects zero drawdown at the stream site. A steady downward recharge causes groundwater to move in the horizontal plane.

2.2. Variational formulation of the problem

The geometry of aquifer renders conventional solution techniques powerless in dealing with intended boundary-value problem. To overcome this difficulty, the variational approach becomes more effective in dealing with domains confined by irregular boundaries. First, the associated penalty function is formed as: