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Simulation of unsteady flow over floodplain using the diffusive wave equation and the modified finite element method

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SUMMARY

We consider solution of 2D nonlinear diffusive wave equation in a domain temporarily covered by a layer of water. A modified finite element method with triangular elements and linear shape functions is used for spatial discretization. The proposed modification refers to the procedure of spatial integration and leads to a more general algorithm involving a weighting parameter. The standard finite element method and the finite difference method are its particular cases. Time integration is performed using a two-stage difference scheme with another weighting parameter. The resulting systems of nonlinear algebraic equations are solved using the Picard and Newton iterative methods. It is shown that the two weighting parameters determine the accuracy and stability of the numerical solution as well as the convergence of iterative process. Accuracy analysis using the modified equation approach carried out for linear version of the governing equation allowed to evaluate the numerical diffusion and dispersion generated by the method as well as to explain its properties.

As the finite element method accounts for the Neumann type of boundary conditions in a natural way, no special treatment of the boundary is needed. Consequently the problem of moving grid point, which must follow the shoreline, in the proposed approach is overcome automatically. The current position of moving boundary is obtained as a result of solution of the governing equation at fixed grid point.

1. Introduction

In hydrological practice one very often faces the problem of delimitation of the flooding large areas adjacent the river, which are initially dry. Typically such a situation arises while analyzing the dam failure or dike break problems. If, for instance, during the flood period the protecting dike is destroyed, the water will flow out of the breach usually covering very large area. For evident reasons it is important to know the extent of the inundation in space and in time. This can be done by solving 2D shallow water equations (Weiyan, 1992), i.e. the continuity equation:

$$\frac{\partial H}{\partial t}+\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}=\Phi \tag{1}$$

and two dynamic equations written in x and y directions as follows:

$$\frac{\partial q_x}{\partial t}+\frac{\partial}{\partial x}\left(\frac{q_x^2}{h}\right)+g\frac{\partial H}{\partial x}+g\frac{n^2}{h^{10/3}}|q|q_x=0 \tag{2}$$

$$\frac{\partial q_y}{\partial t}+\frac{\partial}{\partial y}\left(\frac{q_y^2}{h}\right)+g\frac{\partial H}{\partial y}+g\frac{n^2}{h^{10/3}}|q|q_y=0 \tag{3}$$

where x, y is the space co-ordinates, t the time, H the water surface elevation above the assumed datum, $h = H–Z$ the flow depth, Z the bottom elevation above the assumed datum, q_x the specific discharge in x direction, q_y the specific discharge in y direction, $\Phi = P–E–I$ the source term which usually involves the rainfall (P), evaporation (E) and infiltration (I), n the Manning roughness coefficient, g the acceleration due to gravity, $|q|=\sqrt{q_x^2+q_y^2}$ is the modulus of specific discharge in the flow direction.

Since the flooded area is initially dry, Eqs. (1)–(3) must be integrated over the solution domain, which develops in time. In other words, the shallow water equations are solved over the domain delimited by a moving shoreline, which separates dry and wet areas. Therefore we are looking not only for the solution of mentioned equations within the varying domain but also for the current position of the moving boundaries. Because Eqs. (1)–(3) hold for the flow depth greater than zero, they formally cannot be solved over dry area. When new parts of flooded area are continuously included into or excluded from the solution domain, the numerical solution of the shallow water equations constitutes non-trivial problem. There are various techniques for reproducing moving shoreline, e.g. Heniche et al. (2000). Special numerical treatment is necessary since typically the shoreline is going between the neighboring nodes. Additional difficulty appears when the flow becomes supercritical. It should be added that if in the

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solution of 2D shallow water equations the discontinuities are expected, the robust and reliable approach seems to be the finite volume method (LeVeque, 2002).

Very often the problem of water flow over initially dry area can be solved using the approximate approach provided by simplification of the governing equations. In Eqs. (2) and (3) the inertial force can be neglected. Then we will have the original continuity equation and the simplified dynamic equations. Consequently one obtains a simplified model of unsteady flow in the form of 2D diffusive wave, which can be reduced to a single equation with one unknown function.

There are many reasons which encourage applying such kind of approach. For instance, Horritt and Bates (2001) compared two approaches: a raster based model using the simplified flow equation and 2D full shallow water equations solved with the finite element method (FEM), concluded "…that topography is more important than process representation, and a relatively simple model can be used to good effect". On the other hand Prestininzi (2008) tested the parabolic approximation of the 2D shallow water equations against data from a physical model of dam break event. Their analysis suggests that the parabolic model may effectively reproduce the principal features of an inundation flow even in its extreme case such as the dam break problem. If the simplified models can be applied for a dam beak event one can expect them to be useful also for a dike break. When water enters the floodplain one can suppose that except the vicinity of breach, it is characterized by rather low dynamics. The flooding flow due to dike break propagates freely in a large area so it has rather diffusive character with the forces of gravity, pressure and friction dominating (Zhang et al., 2004). Similar conclusions are presented by other authors (see for example large review given by Singh (1996). In this paper the diffusive wave is considered.

2D diffusive wave equation is obtained via simplification of the system of shallow water equations. Its derivation was proposed by Hromadka and Yen (1986). Simplification of both dynamic equations is carried out by neglecting of the inertial force. Therefore they become as follows:

$$\frac{\partial H}{\partial x} + \frac{n^2}{h^{10/3}}|q|q_x = 0 \tag{4}$$

$$\frac{\partial H}{\partial y} + \frac{n^2}{h^{10/3}}|q|q_y = 0 \tag{5}$$

where $q_x = q \cdot \cos\varphi$ and $q_y = q \cdot \sin\varphi$ (φ is angle between the flow direction and x axis: $\varphi = \arctan(q_x/q_y)$). Using the Manning formula the components of mass flux are expressed as:

$$q_x = \frac{\frac{1}{n}\cdot h^{5/3}}{\left|\frac{\partial H}{\partial s}\right|^{1/2}}\frac{\partial H}{\partial x} = -K_x\frac{\partial H}{\partial x} \tag{6}$$

$$q_y = \frac{\frac{1}{n}\cdot h^{5/3}}{\left|\frac{\partial H}{\partial s}\right|^{1/2}}\frac{\partial H}{\partial y} = -K_y\frac{\partial H}{\partial y} \tag{7}$$

where $\partial H/\partial s$ is a gradient of the water stage with regard to the flow direction (denoted by s). With Eqs. (6) and (7) the continuity Eq. (1) takes the following form:

$$\frac{\partial H}{\partial t} - \frac{\partial}{\partial x}\left(K_x\cdot\frac{\partial H}{\partial x}\right) - \frac{\partial}{\partial y}\left(K_y\cdot\frac{\partial H}{\partial y}\right) = \Phi \tag{8}$$

For details of the derivation see Hromadka and Yen (1986) or Singh (1996).

Eq. (8), known as 2D nonlinear diffusive wave equation, is a partial differential equation of 2nd order of parabolic type with source term. Due its strong nonlinearity this equation seems to be very similar to the Richards equation describing the flow in unsaturated porous media (e.g. Weill et al., 2009). In order to obtain the function $H(x,y,t)$ Eq. (8) must be integrated numerically for initial and boundary conditions properly imposed at the limits of solution domain.

Solution of the diffusive wave equation (8) can be carried out using the Nodal Domain Integration Method (Hromadka and Yen (1986), Singh (1996)). The method is based on the finite difference technique with uniform rectangular meshes. It was also developed for triangular mesh (see for instance Zhang et al. (2004)). The integration in time is commonly carried out using the explicit or implicit formulas of 1st order.

The diffusive wave approach can be also applied in a different way. Instead of Eq. (8) one can use the storage cell equation obtained by space integration of the differential continuity equation and the Manning formula governing the flow between cells. Although this technique is known since many years (Cunge, 1975) it is still applied. For instance recently Moussa and Bocquillon (2009) applied this method for solving the flow problem over floodplain of rather complex river system. The same approach with a modified method for solving the intercell flux was applied by Prestininzi (2008).

As far as the finite element method (FEM) is considered, it is well known that this method is particularly suitable for solving 2D parabolic equations. Although it can be successfully applied for 1D flow and transport equations (Blandford and Ormsbee, 1993; Szymkiewicz, 1995) its advantages are more clear for 2D and 3D problems. The two most valuable features of FEM are: very flexible spatial discretization and simple way of introducing of the imposed boundary conditions. Moreover in the case of diffusive wave equation FEM provides the numerical solution including the position of the boundary on the fixed grid point covering both dry and wet parts of the considered flow area. It is interesting that this method is frequently used for solution of the 2D shallow water equations (Heniche et al. (2000), Horritt and Bates (2001), Horritt (2002)), but it is seldom applied for solution of the 2D nonlinear diffusive wave equation (8). Recently the finite element technique has been used by Weill et al. (2009). One possible explanation might be that the Galerkin FEM generates oscillations in the case of advection dominated flow (Heniche et al., 2000). As a matter of fact these oscillations are caused by the numerical dispersion produced by the method. It appears that in the case of the triangle elements and linear trial functions it is possible to modify the standard FEM in such a way that the dispersion error is remarkably reduced. This improvement of FEM is the subject of this paper.

2. Solution of 2D diffusive wave equation using the modified finite element method

FEM is one of the most popular approaches for solving the partial differential equations. Comprehensive presentations of the method oriented for solving the fluid mechanics problems are given by Gresho and Sani (1998), Fletcher (1991), Oden and Reddy (1976) among others. To better explain the proposed modification of the method let us start with short description of its standard form.

In the finite element method the continuous domain is divided into smaller subareas – the finite elements. It is assumed that these elements are joined in a finite number of points lying at the element's boundary. These points are the nodes, in which the approximate solution will be computed. Let us divide the continuous domain C closed by the boundary B into N triangular elements using M nodes as it is shown in Fig. 1.

According to the Galerkin procedure (Fletcher, 1991; Zienkiewicz, 1972) the solution of Eq. (8) should satisfy the condition: